Applications of the First Derivative

Increasing/Decreasing functions

So far you have only been able to determine if a function is increasing or decreasing by plotting points to graph the function. Now that you know how to find the derivative of a function, you will learn how the derivative can be used to determine the intervals where a function is increasing or decreasing.

Remember the derivative of a function represents the slope of the tangent line at a particular point on the graph. So if the derivative is positive on a open interval, say (a, b), then the slope of the tangent line is positive which means that the function must be increasing on the open interval (a, b). If the derivative is negative on the open interval (a, b) then the function must be decreasing on this interval. If the derivative happens to be zero then the function is constant, neither increasing nor decreasing.

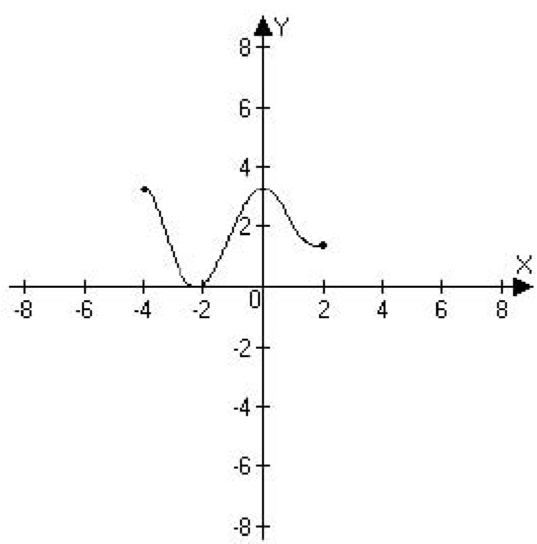
Therefore the definition for increasing and decreasing functions can be described as:

If f is a function defined on the interval (a, b) containing the numbers x_1 and x_2, then

f(x) is increasing on the interval (a, b) if $f(x_1) < f(x_2)$ when $x_1 < x_2$

f(x) is decreasing on the interval (a, b) if $f(x_1) > f(x_2)$ when $x_1 < x_2$

Example 1: In the given graph of the function f(x), determine the interval(s) where the function is increasing, decreasing, or constant.



Example 1 (Continued):

Solution:

Looking at the graph from left to right you would have the following three intervals.

the function is decreasing on the interval (-4, -2)
the function is increasing on the interval (-2, 0)
the function is decreasing on the interval (0, 2)

In order to find the intervals where a function is increasing, decreasing, or constant without first graphing the function, you must find what are called the critical numbers. The critical numbers are those contained in the domain of f(x) and which make the first derivative equal to zero or undefined.

Let's look at an example of how to find the critical numbers. If you are given the function of f as being $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 10$ then the first derivative would be $f'(x) = x^2 - 2x - 3$.

$$f(x) = \frac{1}{3}x^3 - x^2 - 3x + 10$$
$$f'(x) = \frac{1}{3} \cdot 3x^2 - 2x - 3 + 0$$
$$f'(x) = x^2 - 2x - 3$$

Since the derivative is a polynomial function we do not need to worry about it being undefined, so we just need to find out where it is equal to zero. You would do this by setting the derivative equal to zero and then solve for x. With this example you can solve by factoring, however some other derivatives may require you to use other methods such as the quadratic formula or synthetic division.

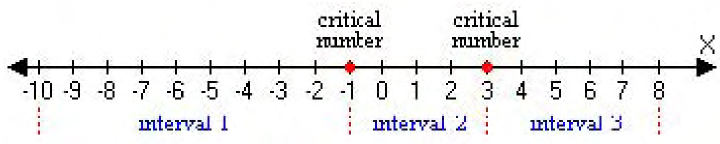
$$f'(x) = x^2 - 2x - 3$$
$$0 = x^2 - 2x - 3$$
$$0 = (x+1)(x-3)$$

$$x + 1 = 0 \qquad x - 3 = 0$$
$$x = -1 \qquad x = 3$$

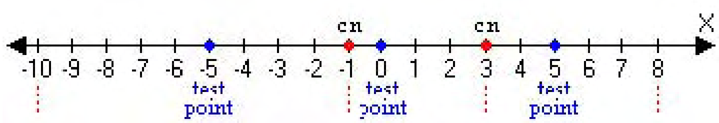
So the critical numbers are –1 and 3.

The critical number(s) will be used to split up the domain of f into intervals. The first derivative would then be evaluated at a test point in each interval to determine whether it is positive (increasing) or negative (decreasing).

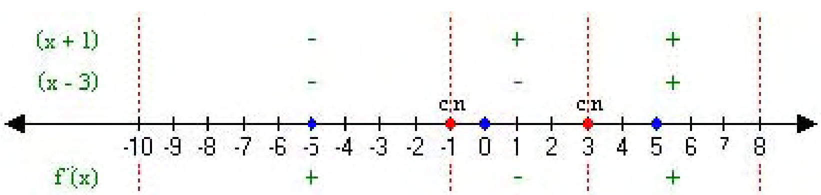
Continuing with our previous example, lets say you are looking for the intervals where f(x) is increasing or decreasing on the open interval (-10, 8). The critical numbers of –1 and 3 will split up the open interval of (-10, 8) into the following three intervals (-10, -1), (-1, 3), and (3, 8).



You would now choose a number in each of these three intervals with which to evaluate the first derivative. When choosing a test point, you can pick any number you want to use within each interval. However, whenever possible choose zero as one of the test points since it is the easiest point at which to evaluate the function. In the following example, we choose –5, 0, and 5 as the test points.



When we evaluate the derivative with the test points we are only concerned about what the sign (positive or negative) the number does not matter. So one way to determine the sign of the derivative in each interval is to list all of the factors above the number line and determine their signs at each test point. Once all the signs of all of the factors have been determined, you can multiply them to find the sign of the derivative. Remember that when multiplying negative signs, an odd number of negatives will yield a negative product and an even number of negatives will yield a positive product. Let's continue to use our previous example to see how this process would be done.



Now that you know the signs of the derivative in each interval you can see where it is increasing (+) or decreasing (-).

f(x) is increasing on the intervals (-10, -1) and (3, 8)
f(x) is decreasing on the interval (-1, 3)

Test for intervals where f(x) is increasing or decreasing

If f is a function with a derivative at all points in the open interval (a, b) and "p" is a test point in this interval, then

f is increasing on the interval (a, b) if $f'(p) > 0$
f is decreasing on the interval (a, b) if $f'(p) < 0$
f is constant on the interval (a, b) if $f'(p) = 0$

The steps to follow in determining where a function is increasing or decreasing can be summarized as follows:

1. Find the derivative of the given function.
2. Locate any critical numbers by seeing where the derivative is either zero or undefined.
3. Plot the critical numbers on a number line to determine the open intervals.
4. Select a test point in each interval and evaluate the derivative at this point.
5. Use the sign of the derivative in each interval to determine whether it is increasing or decreasing.

Example 2: For the function $f(x) = x^4 - \frac{4}{3}x^3 - 4x^2 + 1$ determine the intervals where it is increasing or decreasing.

Solution:

Step 1: Find the derivative of the given function.

$$f(x) = x^4 - \frac{4}{3}x^3 - 4x^2 + 1$$
$$f'(x) = 4x^{4-1} - \frac{4}{3}\cdot 3x^{3-1} - 4\cdot 2x^{2-1} + 0$$
$$f'(x) = 4x^3 - 4x^2 - 8x$$

Example 2 (Continued):

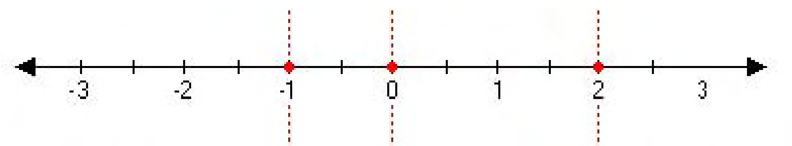
Step 2: Locate any critical numbers.

Since the derivative is a polynomial function you only need to find where it is equal to zero. To do this, you will need to factor the derivative and then set each factor equal to zero.

$$f'(x) = 4x^3 - 4x^2 - 8x$$
$$= 4x(x^2 - x - 2)$$
$$= 4x(x+1)(x-2)$$

$$4x = 0 \qquad x+1=0 \qquad x-2=0$$
$$x = 0 \qquad x = -1 \qquad x = 2$$

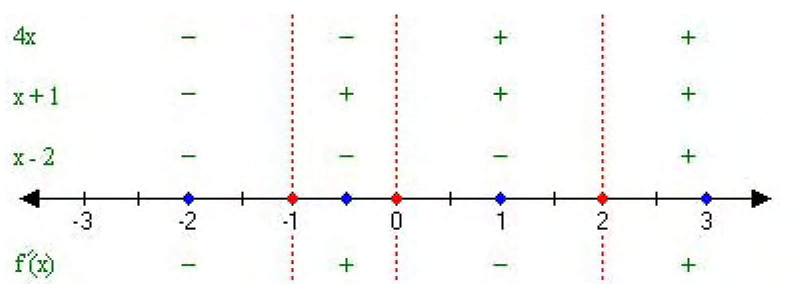
Step 3: Determine the intervals to be tested.



Since there are three critical numbers you will have four intervals

$$(-\infty,-1),\ (-1,0),\ (0,2),\ \textit{and}\ (2,\infty)$$

Step 4: Select a test point in each interval and evaluate the sign of the derivative.



Example 2 (Continued):

Step 5: Identify the intervals that are increasing or decreasing.

f is increasing on the intervals (-1, 0) and (2, ∞)
f is decreasing on the intervals (-∞, -1) and (0, 2)

Example 3: Determine the intervals where $f(x) = \dfrac{x^2+4}{2x+1}$ is increasing or decreasing.

Solution:

Step 1: Find the derivative.

$$f(x) = \frac{x^2+4}{2x+1}$$

$$f'(x) = \frac{(2x+1)D_x(x^2+4) - (x^2+4)D_x(2x+1)}{(2x+1)^2}$$

$$= \frac{(2x+1)(2x) - (x^2+4)(2)}{(2x+1)^2}$$

$$= \frac{(4x^2+2x) - (2x^2+8)}{(2x+1)^2}$$

$$= \frac{4x^2+2x-2x^2-8}{(2x+1)^2}$$

$$= \frac{2x^2+2x-8}{(2x+1)^2}$$

Step 2: Locate any critical numbers.

In this example, the derivative is a rational function. Therefore, to find the critical numbers we must look at where the derivative is zero and where it is undefined. The derivative will be zero when the numerator is zero and it will be undefined when the denominator is zero.

Example 3 (Continued):

$$2x^2 + 2x - 8 = 0$$
$$x^2 + x - 4 = 0$$

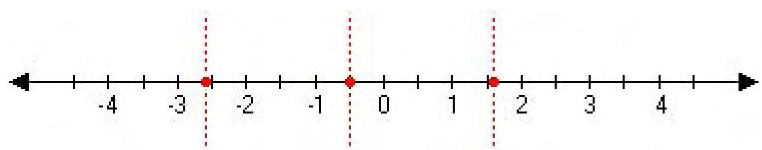
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(-4)}}{2(1)}$$
$$= \frac{-1 \pm \sqrt{1+16}}{2}$$
$$= \frac{-1 \pm \sqrt{17}}{2}$$

$$(2x+1)^2 = 0$$
$$2x + 1 = 0$$
$$2x = -1$$
$$x = -\frac{1}{2}$$

The critical numbers are $\frac{-1-\sqrt{17}}{2}, -\frac{1}{2}, \text{ and } \frac{-1+\sqrt{17}}{2}$. .

Step 3: Determine the intervals to be tested.

$\frac{-1-\sqrt{17}}{2} \approx -2.56$ and $\frac{-1+\sqrt{17}}{2} \approx 1.56$

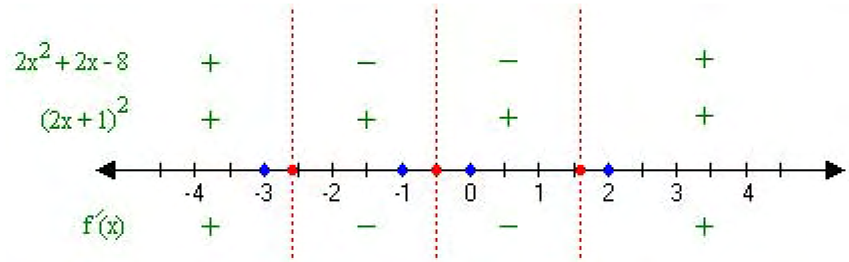


The four intervals will be:

$$\left(-\infty, \frac{-1-\sqrt{17}}{2}\right), \left(\frac{-1-\sqrt{17}}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{-1+\sqrt{17}}{2}\right), \text{ and } \left(\frac{-1+\sqrt{17}}{2}, \infty\right)$$

Example 3 (Continued):

Step 4: Select a test points and evaluate the derivative.



Step 5: Identify the intervals that are increasing or decreasing.

f is increasing on the intervals $\left(-\infty, \frac{-1-\sqrt{17}}{2}\right)$ and $\left(\frac{-1+\sqrt{17}}{2}, \infty\right)$

f is decreasing on the intervals $\left(\frac{-1-\sqrt{17}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{-1+\sqrt{17}}{2}\right)$

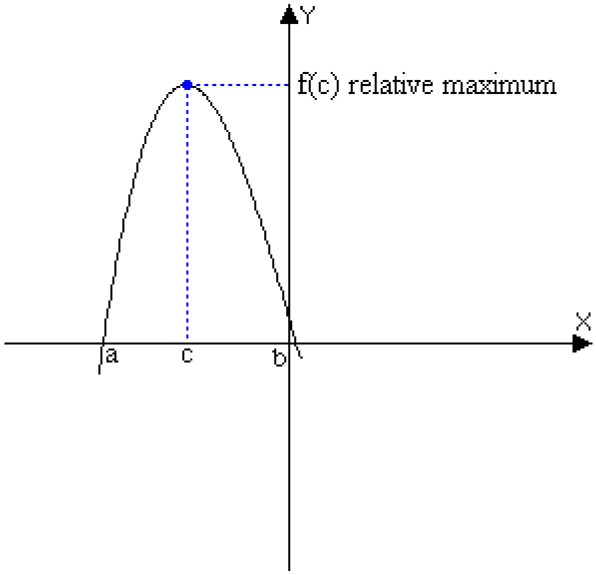
Relative Extrema

The relative extrema of a function are the points where a relative (local) maximum or minimum point exists in the open interval (a, b). When locating the relative extrema you will want to look at the critical numbers derived from the first derivative and any endpoints of the function.

A relative maximum would be the highest point in the open interval (a, b). Therefore, the value of the function at the relative maximum point should be greater than the value of the function at all other points in the same open interval.

If c is a number located in the open interval of (a, b) and is included in the domain of the function f, then a relative maximum exists at f(c) when

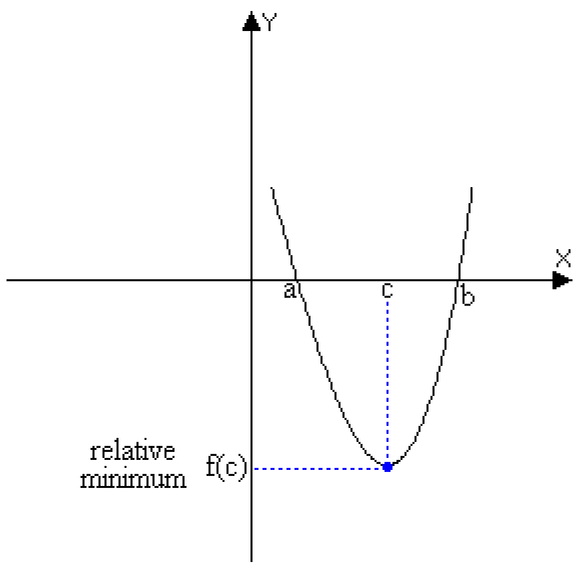
$f(x) \le f(c)$ for all x in the open interval (a, b)



A relative minimum on the other hand would be the lowest point in the open interval (a, b).

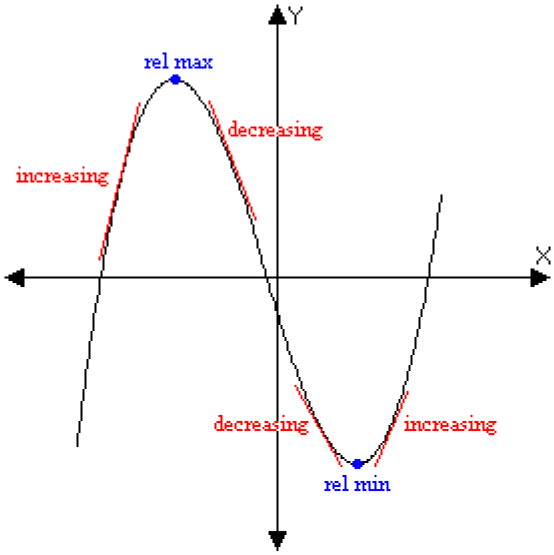
If c is a number located in the open interval of (a, b) and is included in the domain of the function f, then a relative minimum exists at f(c) when

$f(x) \geq f(c)$ for all x in the open interval (a, b)



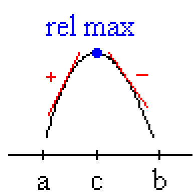
In the last section, you learned how to locate the critical numbers of a function and use them to find the intervals where the function is increasing or decreasing. The slopes of the tangent lines can also be used to determine where any relative extrema exist.

Let's take a look at the graphs used to illustrate relative maximums and minimums, but this time also show the tangent lines and whether the function is increasing or decreasing.



Remembering from last section, an interval is said to be increasing if the slope of the tangent line is positive and decreasing if negative. Combining this with what you know about relative extrema will give you the basis of the first derivative test, which can be used for determining the intervals that are increasing/decreasing and for finding any relative extrema.

First derivative test

> Let c be a critical number for a function f. If f is differentiable on the open interval (a, b), except possibly at c, and c is the only critical number in this interval then:
>
> 1. f(c) is a relative maximum if the derivative changes sign from positive to negative on the intervals (a, c) and (c, b).
>
> 
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>
> 2. f(c) is a relative minimum if the derivative changes sign from negative to positive on the intervals (a, c) and (c, b).
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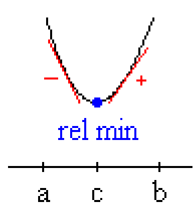
> 
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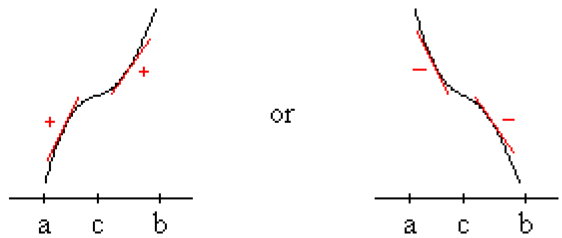
3. f(c) is neither a relative maximum or minimum if the derivative remains the same sign on both intervals (a, c) and (c, b).



When asked to find the relative extrema you would begin with the same steps used in the last section for determining the intervals that are increasing or decreasing. Once you know the signs of the intervals you would then identify the relative extrema based on the first derivative test.

1. Find the derivative of the given function.
2. Locate any critical numbers by seeing where the derivative is either zero or undefined.
3. Plot the critical numbers on a number line to determine the open intervals.
4. Select a test point in each interval and evaluate the derivative at this point.
5. Use the sign of the derivative in each interval to determine whether it is increasing or decreasing.
6. Identify the relative extrema based on the sign changes.

Example 4: Find the intervals where the function is increasing or decreasing and all relative extrema.

$$f(x) = 2x^3 - 3x^2 - 12x + 1$$

Solution:

Step 1: Find the first derivative.

$$f(x) = 2x^3 - 3x^2 - 12x + 1$$
$$f'(x) = 2(3x^2) - 3(2x) - 12(1) + 0$$
$$= 6x^2 - 6x - 12$$

Example 4 (Continued):

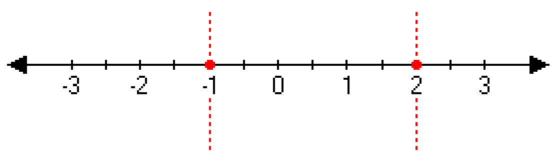
Step 2: Find the critical numbers.

$$f'(x) = 6x^2 - 6x - 12$$
$$0 = 6(x^2 - x - 2)$$
$$0 = 6(x-2)(x+1)$$

$$x - 2 = 0 \qquad x + 1 = 0$$
$$x = 2 \qquad x = -1$$

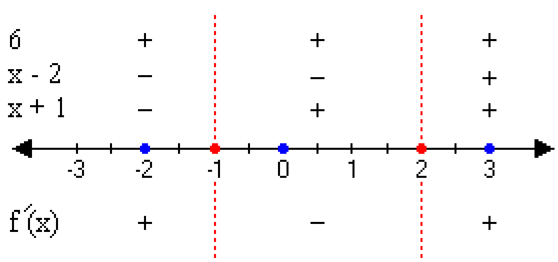
The critical numbers are –1 and 2.

Step 3: Plot critical numbers to determine the intervals.



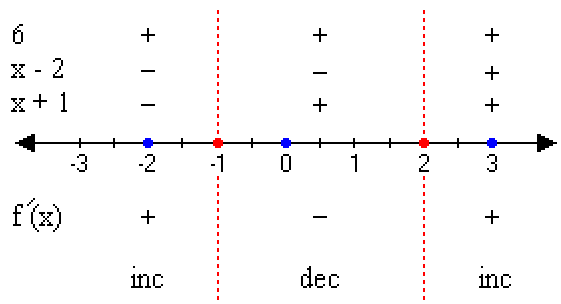
The intervals are $(-\infty,-1), (-1,2),\ and\ (2,\infty)$

Step 4: Select test points and evaluate the derivative.



Example 4 (Continued):

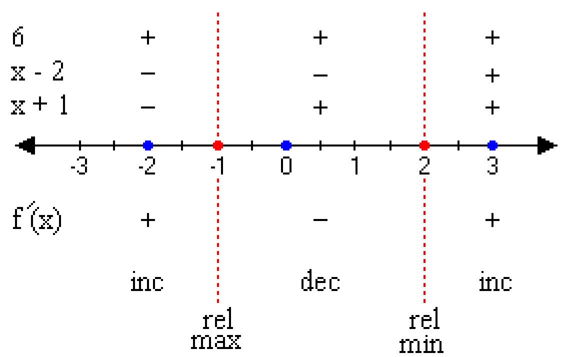
Step 5: Identify intervals that are increasing/decreasing.



f(x) is increasing on the intervals $(-\infty,-1)$ *and* $(2,\infty)$

f(x) is decreasing on the interval $(-1,2)$

Step 6: Identify any relative extrema.



$$\begin{aligned} f(x) &= 2x^3 - 3x^2 - 12x + 1 \\ f(-1) &= 2(-1)^3 - 3(-1)^2 - 12(-1) + 1 \\ &= 2(-1) - 3(1) + 12 + 1 \\ &= -2 - 3 + 13 \\ &= 8 \end{aligned}$$

The relative maximum would be at (-1, 8)

Example 4 (Continued):

$$\begin{aligned} f(x) &= 2x^3 - 3x^2 - 12x + 1 \\ f(2) &= 2(2)^3 - 3(2)^2 - 12(2) + 1 \\ &= 2(8) - 3(4) - 24 + 1 \\ &= 16 - 12 - 23 \\ &= -19 \end{aligned}$$

The relative minimum would be at (2, -19)

Example 5: Find the intervals where the function is increasing/decreasing and the relative extrema.

$$g(x) = \frac{xe^x}{x-1}$$

Solution:

Step 1: Find the first derivative.

To find the derivative you will need to use the quotient rule, product rule, and the exponential rule

$$\begin{aligned} g(x) &= \frac{xe^x}{x-1} \\ g'(x) &= \frac{(x-1)D_x(xe^x) - (xe^x)D_x(x-1)}{(x-1)^2} \\ &= \frac{(x-1)\left[(x)D_x(e^x) + (e^x)D_x(x)\right] - (xe^x)(1)}{(x-1)^2} \\ &= \frac{(x-1)\left[(x)(e^x) + (e^x)(1)\right] - (xe^x)}{(x-1)^2} \\ &= \frac{(x-1)(xe^x + e^x) - (xe^x)}{(x-1)^2} \end{aligned}$$

Example 5 (Continued):

$$g'(x) = \frac{(x-1)(e^x)(x+1) - (xe^x)}{(x-1)^2}$$

$$= \frac{(e^x)\left[(x-1)(x+1) - x\right]}{(x-1)^2}$$

$$= \frac{(e^x)\left[(x^2-1) - x\right]}{(x-1)^2}$$

$$= \frac{(e^x)(x^2-x-1)}{(x-1)^2}$$

Step 2: Find the critical numbers.

Recall that the critical numbers will occur where the derivative is either zero or undefined. In order for the derivative to be equal to zero the numerator must be zero. However, the exponential function e^x will never be zero so in the numerator we will only look at the polynomial factor of $x^2 - x - 1$.

$$(e^x)(x^2 - x - 1) = 0$$

$$x^2 - x - 1 = 0$$

The polynomial, $x^2 - x - 1$, cannot be factored so you can use the quadratic formula to solve for x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

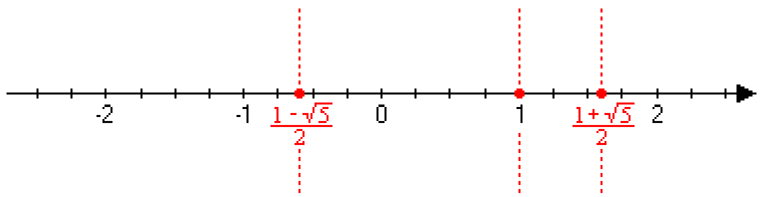
Example 5 (Continued):

Now you must determine where the derivative will be undefined. Since the derivative is a fraction, it will be undefined if the denominator is zero. Therefore, you will now let the denominator be equal to zero and solve for x.

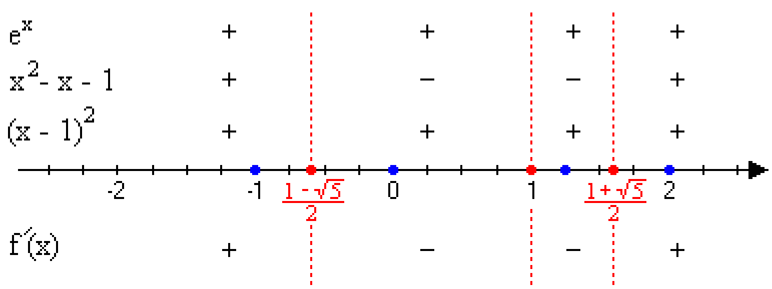
$$(x-1)^2 = 0$$
$$x-1=0$$
$$x=1$$

The critical numbers are $\frac{1-\sqrt{5}}{2} \approx -0.6,\ 1,\ \textit{and}\ \frac{1+\sqrt{5}}{2} \approx 1.6$

Step 3: Plot critical numbers to determine the intervals.



Step 4: Select test points and evaluate the derivative.



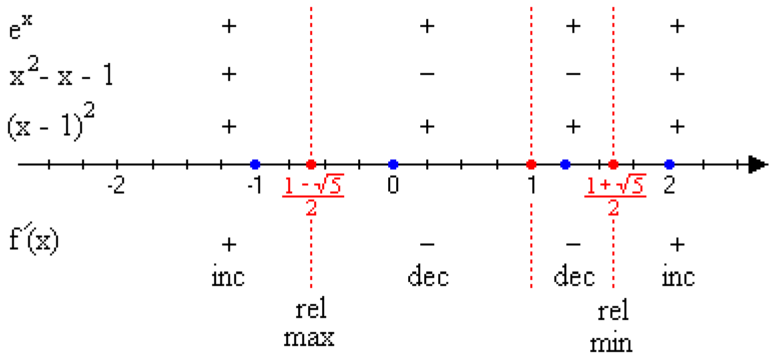
Example 5 (Continued):

Step 5: Identify intervals that are increasing/decreasing.

f(x) is increasing on the intervals $\left(-\infty, \frac{1-\sqrt{5}}{2}\right)$ *and* $\left(\frac{1+\sqrt{5}}{2}, \infty\right)$

f(x) is decreasing on the intervals $\left(\frac{1-\sqrt{5}}{2}, 1\right)$ *and* $\left(1, \frac{1+\sqrt{5}}{2}\right)$

Step 6: Identify any relative extrema.



$$g(x) = \frac{xe^x}{x-1}$$

$$g\left(\frac{1-\sqrt{5}}{2}\right) = \frac{\left(\frac{1-\sqrt{5}}{2}\right)e^{\left(\frac{1-\sqrt{5}}{2}\right)}}{\left(\frac{1-\sqrt{5}}{2}\right)-1}$$

$$\approx \frac{(-0.618)e^{(-0.618)}}{(-0.618)-1}$$

$$\approx \frac{(-0.618)(0.539)}{-1.618}$$

$$\approx \frac{-0.333}{-1.618}$$

$$\approx 0.206$$

The relative maximum is approximately at (-0.618, 0.206).

Example 5 (Continued):

$$g(x) = \frac{xe^x}{x-1}$$

$$g\left(\frac{1+\sqrt{5}}{2}\right) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)e^{\left(\frac{1+\sqrt{5}}{2}\right)}}{\left(\frac{1+\sqrt{5}}{2}\right)-1}$$

$$\approx \frac{(1.618)e^{(1.618)}}{(1.618)-1}$$

$$\approx \frac{(1.618)(5.043)}{0.618}$$

$$\approx \frac{8.160}{0.618}$$

$$\approx 13.204$$

The relative minimum is approximately at (1.618, 13.204).