Applications of the Second Derivative

Concavity

Concavity of a function can be determined by the use of the second derivative. In the previous section you saw how the first derivative was used to determine where the function was increasing or decreasing.

The process of finding the intervals where a function is concave up or concave down is very similar. Once you have the second derivative, you will set it equal to zero to find your critical points. These critical points will setup the intervals that you will test for concavity.

The function is concave upward if the second derivative is positive on an open interval (a, b) and concave downward when the second derivative is negative on an open interval (a, b). A point of inflection will occur whenever there is a change in the function's concavity.

Test for concavity

If f(x) is a function with first and second derivatives defined for all points in the open interval (a, b), then:

f(x) is concave upward if $f''(x) > 0$ for all x in the interval (a, b)

f(x) is concave downward if $f''(x) < 0$ for all x in the interval (a, b)

Example 1: Find the intervals where f(x) = $3x^4 - 5x^2 - 11x$ is concave upward or downward, and find any inflection points.

Solution:

Step 1: Find the first derivative.

$$\begin{aligned} f(x) &= 3x^4 - 5x^2 - 11x \\ f'(x) &= 3\left(4x^{4-1}\right) - 5\left(2x^{2-1}\right) - 11\left(1x^{1-1}\right) \\ &= 3\left(4x^3\right) - 5\left(2x\right) - 11\left(1\right) \\ &= 12x^3 - 10x - 11 \end{aligned}$$

Example 1 (Continued):

Step 2: Find the second derivative.

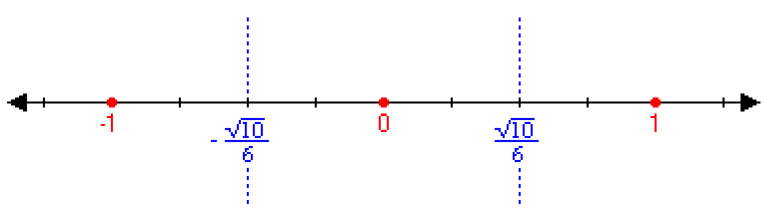
$$f'(x) = 12x^3 - 10x - 11$$
$$f''(x) = 12\left(3x^{3-1}\right) - 10\left(1x^{1-1}\right) - 0$$
$$= 12\left(3x^2\right) - 10(1)$$
$$= 36x^2 - 10$$

Step 3: Find the critical numbers of the second derivative.

$$f''(x) = 36x^2 - 10$$
$$0 = 36x^2 - 10$$
$$10 = 36x^2$$
$$\frac{10}{36} = x^2$$
$$\sqrt{\frac{10}{36}} = \sqrt{x^2}$$
$$\frac{\pm\sqrt{10}}{6} = x$$

$$x = \frac{\sqrt{10}}{6} \qquad x = -\frac{\sqrt{10}}{6}$$
$$\approx 0.5 \qquad \approx -0.5$$

Step 4: Graph the critical numbers on a number line and select test points for each interval.



Example 1 (Continued):

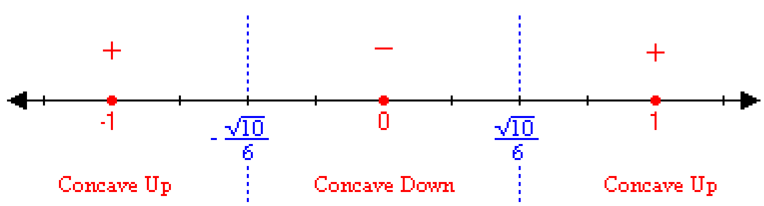
For the interval $\left(-\infty, -\frac{\sqrt{10}}{6}\right)$ we will use –1 as the test point

For the interval $\left(-\frac{\sqrt{10}}{6}, \frac{\sqrt{10}}{6}\right)$ we will use 0 as the test point

For the interval $\left(\frac{\sqrt{10}}{6}, \infty\right)$ we will use 1 as the test point

Step 5: Determine the concavity for each interval

$$\begin{aligned} f''(x) &= 36x^2 - 10 \\ f''(-1) &= 36(-1)^2 - 10 \\ &= 36(1) - 10 \\ &= 26 \end{aligned} \qquad \begin{aligned} f''(x) &= 36x^2 - 10 \\ f''(0) &= 36(0)^2 - 10 \\ &= 36(0) - 10 \\ &= -10 \end{aligned} \qquad \begin{aligned} f''(x) &= 36x^2 - 10 \\ f''(1) &= 36(1)^2 - 10 \\ &= 36(1) - 10 \\ &= 26 \end{aligned}$$



The function is concave up on the intervals $\left(-\infty, -\frac{\sqrt{10}}{6}\right)$ and $\left(\frac{\sqrt{10}}{6}, \infty\right)$

The function is concave down on the interval $\left(-\frac{\sqrt{10}}{6}, \frac{\sqrt{10}}{6}\right)$

Example 1 (Continued):

Step 6: Locate any points of inflection.

$$f(x) = 3x^4 - 5x^2 - 11x$$

$$f\left(-\frac{\sqrt{10}}{6}\right) = 3\left(-\frac{\sqrt{10}}{6}\right)^4 - 5\left(-\frac{\sqrt{10}}{6}\right)^2 - 11\left(-\frac{\sqrt{10}}{6}\right)$$

$$= 3\left(\frac{100}{1296}\right) - 5\left(\frac{10}{36}\right) - 11\left(-\frac{\sqrt{10}}{6}\right)$$

$$= \frac{25}{108} - \frac{50}{36} + \frac{11\sqrt{10}}{6}$$

$$= \frac{25}{108} - \frac{150}{108} + \frac{198\sqrt{10}}{108}$$

$$= -\frac{125}{108} + \frac{198\sqrt{10}}{108}$$

$$= \frac{-125 + 198\sqrt{10}}{108}$$

$$\approx -4.64$$

The point of inflection would be approximately at (-0.5, -4.64)

$$f(x) = 3x^4 - 5x^2 - 11x$$

$$f\left(\frac{\sqrt{10}}{6}\right) = 3\left(\frac{\sqrt{10}}{6}\right)^4 - 5\left(\frac{\sqrt{10}}{6}\right)^2 - 11\left(\frac{\sqrt{10}}{6}\right)$$

$$= 3\left(\frac{100}{1296}\right) - 5\left(\frac{10}{36}\right) - \frac{11\sqrt{10}}{6}$$

$$= \frac{25}{108} - \frac{50}{36} - \frac{11\sqrt{10}}{6}$$

$$= \frac{25}{108} - \frac{150}{108} - \frac{198\sqrt{10}}{108}$$

$$= -\frac{125}{108} - \frac{198\sqrt{10}}{108}$$

$$= \frac{-125 - 198\sqrt{10}}{108}$$

$$\approx -6.95$$

The point of inflection would be approximately at (0.5, -6.95)

In some situations, the second derivative can be used to determine where the relative extrema are located. This would be done by using the second derivative test. For the second derivative test you must first find the critical numbers for the first derivative and then evaluate the second derivative at these points. The concavity will indicate whether the points are relative maximums or relative minimums.

Second Derivative Test

If c is a critical number of the first derivative for a given function, then:

f(c) is a relative maximum if $f''(c) > 0$

f(c) is a relative minimum if $f''(c) < 0$

If $f''(c) = 0$ then the second derivative test fails and you must use the first derivative test to determine the relative extrema.

Example 2: Find the relative extrema for the function $f(x) = 3x^3 - 3x^2 + 1$ using the second derivative test. If the second derivative test fails, use the first derivative.

Solution:

Step 1: Find the first derivative.

$$\begin{aligned} f(x) &= 3x^3 - 3x^2 + 1 \\ f'(x) &= 3\left(3x^2\right) - 3\left(2x\right) + 0 \\ &= 9x^2 - 6x \end{aligned}$$

Step 2: Find the critical numbers of the first derivative.

$$\begin{aligned} f'(x) &= 9x^2 - 6x \\ 0 &= 9x^2 - 6x \\ 0 &= 3x(3x - 2) \end{aligned}$$

$$\begin{aligned} 3x - 2 &= 0 \qquad & 3x &= 0 \\ 3x &= 2 & x &= 0 \\ x &= \frac{2}{3} \end{aligned}$$

The critical numbers are $0 \; and \; \frac{2}{3}$

Example 2 (Continued):

Step 3: Find the second derivative.

$$f'(x) = 9x^2 - 6x$$
$$f''(x) = 9(2x) - 6(1)$$
$$= 18x - 6$$

Step 4: Evaluate the second derivative at the first derivative critical numbers.

$$f''(x) = 18x - 6$$
$$f''(0) = 18(0) - 6$$
$$= -6$$

$$f''(x) = 18x - 6$$
$$f''\left(\frac{2}{3}\right) = 18\left(\frac{2}{3}\right) - 6$$
$$= 12 - 6$$
$$= 6$$

$f''(0) < 0$ therefore a relative maximum occurs at f(0)

$f''\left(\frac{2}{3}\right) > 0$ therefore a relative minimum occurs at $f\left(\frac{2}{3}\right)$

Step 5: Find the relative maximum and minimum points.

$$f(x) = 3x^3 - 3x^2 + 1$$
$$f(0) = 3(0)^3 - 3(0)^2 + 1$$
$$= 0 - 0 + 1$$
$$= 1$$

$$f(x) = 3x^3 - 3x^2 + 1$$
$$f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^3 - 3\left(\frac{2}{3}\right)^2 + 1$$
$$= 3\left(\frac{8}{27}\right) - 3\left(\frac{4}{9}\right) + 1$$
$$= \frac{8}{9} - \frac{12}{9} + \frac{9}{9}$$
$$= \frac{5}{9}$$

(0, 1) is the relative maximum

$\left(\frac{2}{3}, \frac{5}{9}\right)$ is the relative minimum