Curve Sketching

So far you have learned how to find 1st and 2nd derivatives of functions and use these derivatives to determine where a function is:

1. Increasing/decreasing
2. Relative extrema
3. Concavity
4. Points of Inflection

We will now look at how all of this information, in addition to some concepts used in college algebra, can be used to assist in sketching the graph of the function.

The process of curve sketching can be performed in the following steps:

1. Determine the domain of the function (look for any restrictions)
2. Find the x- and y-intercepts for the function.
3. Locate any asymptotes (vertical, horizontal, oblique) for the function.
4. Find the 1st derivative of the function
5. Locate any critical numbers for the 1st derivative
6. Determine the intervals where the function is increasing or decreasing
7. Locate any relative extrema
8. Find the 2nd derivative
9. Locate any critical numbers for the 2nd derivative
10. Determine the intervals where the function is concave up or concave down
11. Locate any points of inflection
12. Plot all points (intercepts, relative extrema, inflection points) and asymptotes on the graph. Connect the points with a smooth curve

Before going over an example of using these curve sketching steps, lets first review a few of the college algebra steps that you may have forgotten. When determining the domain of functions there are several common restrictions that you will want to look for. These include:

1. Division by zero

   Example: $\dfrac{1}{x}$ Restriction: $x \neq 0$

2. Taking the square (or other even) root of a negative number

   Example: $\sqrt[n]{x}\,;\textit{where n is an even number}$ Restriction: $x \geq 0$

3. Taking the logarithm of zero or a negative number

Example: $\log_b x;\ b > 0\ but \neq 1$ Restriction: $x > 0$

Example 1: Find the domain for the function $f(x) = \dfrac{x-1}{2x+1}$

Solution:

In this function you have a fraction so you must make sure the denominator is not zero (division by zero).

$$\begin{aligned} 2x+1 &\neq 0 \\ 2x &\neq -1 \\ x &\neq -\frac{1}{2} \end{aligned}$$

The domain of f(x) is all real numbers except for –1/2.

$$D_{f(x)} = \left\{ x \mid x \neq -\frac{1}{2} \right\}$$

When asked to find the x- and y-intercepts you will be substituting zero for either x or y. First, lets look at finding the y-intercept since it normally requires less work. The y-intercept is found by substituting the variable x with zero and then simplifying the expression.

$$\begin{aligned} f(x) &= \frac{4-3x}{x+1} \\ f(0) &= \frac{4-3(0)}{0+1} \\ &= \frac{4}{1} \\ &= 4 \end{aligned}$$

The y-intercept for this function would be at (0, 4)

Finding the x-intercept would be done by now letting y of f(x) be equal to zero and then solve the expression for x. This will often require you to use techniques such as factoring or synthetic division. Lets use the same function as above.

This time we will let f(x) = 0

$$f(x) = \frac{4-3x}{x+1}$$
$$0 = \frac{4-3x}{x+1}$$

Now in order for this fraction to be equal to zero, the numerator (4 – 3x) must be equal to zero. So you can now set just the numerator equal to zero and solve for x.

$$4-3x = 0$$
$$-3x = -4$$
$$x = \frac{4}{3}$$

The x-intercept for this function would be at $\left(\frac{4}{3}, 0\right)$.

Example 2: Find the x- and y-intercepts of the function $f(x) = \frac{x^2-9}{x+1}$.

Solution:

Find the y-intercept by letting x = 0.

$$f(x) = \frac{x^2-9}{x+1}$$
$$f(0) = \frac{(0)^2-9}{0+1}$$
$$= \frac{-9}{1}$$
$$= -9$$

The y-intercept is (0, –9).

Example 2 (Continued):

Find the x-intercept(s) by letting f(x) = 0.

$$f(x) = \frac{x^2 - 9}{x+1}$$
$$0 = \frac{x^2 - 9}{x+1}$$

Now let the numerator equal zero.

$$x^2 - 9 = 0$$
$$x^2 = 9$$
$$\sqrt{x^2} = \sqrt{9}$$
$$x = |3|$$
$$x = \pm 3$$

The x-intercepts are (-3, 0) and (3, 0).

The last concept to review is finding the asymptotes of a function. A rational function can have vertical, horizontal, or oblique asymptotes. An important thing to remember is that the function cannot have both a horizontal and an oblique asymptote. The rational function will have a horizontal asymptote, an oblique asymptote, or neither.

Lets begin with vertical asymptotes. Vertical asymptotes for rational functions occur whenever the denominator is equal to zero. So to find the vertical asymptote you will set just the denominator equal to zero and then solve for x.

Example 3: Find the vertical asymptote of the function $f(x) = \frac{3x+2}{x-4}$.

Solution:

Set the denominator equal to zero and solve for x.

x – 4 = 0
x = 4

The vertical asymptote is x = 4.

Horizontal asymptotes of a rational function can be determined by either using limits as x approaches positive and negative infinity or by comparing the highest degree powers of the numerator and denominator. The degree powers of the rational function can be used to find the horizontal asymptote by following these rules:

Let the rational function be represented by $f(x) = \frac{ax^n}{bx^m}$

If n < m, then the horizontal asymptote will be equal to the x-axis.

y = 0

If n = m, then the horizontal asymptote will be equal to the ratio of the coefficients.

$$y = \frac{a}{b}$$

If n > m, then there is no horizontal asymptote and you must check for an oblique asymptote.

Example 4: Find the horizontal asymptote of the function $f(x) = \frac{6x^3 - x}{x^4 + 2x^2}$.

Solution:

Locate the terms with the highest degree power in both the numerator and denominator.

$$f(x) = \frac{6x^3 - x}{x^4 + 2x^2}$$

Compare the degree powers and apply the rules for horizontal asymptotes.

The degree power of the numerator (n) = 3
The degree power of the denominator (m) = 4

n < m, therefore the horizontal asymptote is at y = 0.

The last asymptote we will discuss is the oblique asymptote. These asymptotes occur for rational functions whenever the degree power of the numerator is one degree larger than the degree power of the denominator. We will use long division to find the equation of the oblique asymptote.

Example 5: Find the oblique asymptote of the rational function $f\left(x\right)=\dfrac{x^2-9}{x+1}$.

Solution:

The degree power of the numerator (2) is one more than the degree power of the denominator (1), so we will have an oblique asymptote.

$$
\begin{array}{r}
x-1 \\
x+1\overline{\big)\,x^2+0x-9} \\
\underline{-\left(x^2+1x\right)} \\
-1x-9 \\
\underline{-\left(-1x-1\right)} \\
-8
\end{array}
$$

The oblique asymptote is y = x – 1.

Once you know where the function is increasing or decreasing and its concavity, you can begin to sketch the graph by looking at the four different possible graphing combinations.

$f'(x)$	+	−	−	+
$f''(x)$	−	−	+	+
Shape of graph				
	increasing concave down	decreasing concave down	decreasing concave up	increasing concave up

Ok, now lets look at an example, which incorporates all of the steps for curve sketching.

Example 6: Graph the function $f(x) = \frac{-4x}{1+2x}$ using the steps for curve sketching.

Solution:

Step 1: Find the domain of f(x).

Since are function is a rational function, the denominator must not be equal to 0.

$$\begin{aligned} 1+2x &\neq 0 \\ 2x &\neq -1 \\ x &\neq -\frac{1}{2} \end{aligned}$$

The domain of f(x) is equal to all real numbers except for –1/2.

$$D_{f(x)} = \left\{ x \mid x \neq -\frac{1}{2} \right\}$$

Step 2: Find the x- and y-intercepts.

Set x = 0 to find y-intercept.

$$\begin{aligned} f(x) &= \frac{-4x}{1+2x} \\ f(0) &= \frac{-4(0)}{1+2(0)} \\ &= \frac{0}{1+0} \\ &= 0 \end{aligned}$$

The y-intercept is at the origin (0, 0).

Example 6 (Continued):

Step 2: Find the x- and y-intercepts.

Set f(x) = 0 to find the x-intercept(s).

$$f(x) = \frac{-4x}{1+2x}$$
$$0 = \frac{-4x}{1+2x}$$
$$0 = -4x$$
$$0 = x$$

The x-intercept is also at the origin (0, 0).

Step 3: Locate any asymptotes.

Set denominator equal to zero to find vertical asymptotes.

$$1+2x = 0$$
$$2x = -1$$
$$x = -\frac{1}{2}$$

The vertical asymptote is $x = -\frac{1}{2}$.

Compare degree powers of the numerator and denominator to find horizontal asymptote, if any.

Degree power of numerator: 1
Degree power of denominator: 1

Degree powers are equal so the horizontal asymptote is equal to the ratio of the coefficients.

$$y = \frac{-4}{2}$$
$$y = -2$$

The horizontal asymptote is at y = -2. Since there is a horizontal asymptote you do not need to look for an oblique asymptote.

Example 6 (Continued):

Step 4: Find the 1st derivative.

$$
\begin{aligned}
f(x) &= \frac{-4x}{1+2x} \\
f'(x) &= \frac{(1+2x)D_x(-4x)-(-4x)D_x(1+2x)}{(1+2x)^2} \\
&= \frac{(1+2x)(-4)-(-4x)(2)}{(1+2x)^2} \\
&= \frac{-4-8x+8x}{(1+2x)^2} \\
&= \frac{-4}{(1+2x)^2}
\end{aligned}
$$

Step 5: Locate the critical numbers of the 1st derivative.

Since the numerator of the derivative is a constant the derivative will never be equal to zero. So the only critical numbers for the 1st derivative will occur when the derivative is undefined (denominator is zero).

$$
\begin{aligned}
(1+2x)^2 &= 0 \\
1+2x &= 0 \\
2x &= -1 \\
x &= -\frac{1}{2}
\end{aligned}
$$

The denominator will only be zero if x is –1/2. However, this value is not included in the domain of the function and is also where our vertical asymptote is located. Therefore, we have no critical numbers for the 1st derivative.

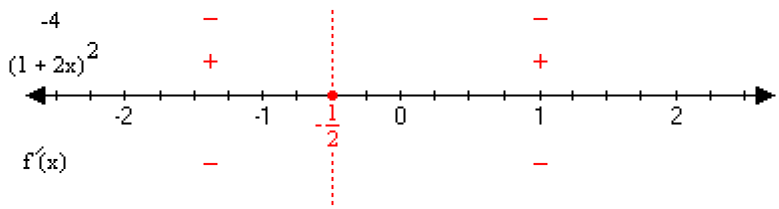
Example 6 (Continued):

Step 6: Determine the intervals where the function is increasing/decreasing.

For this function we have only two intervals to look at

$$\left(-\infty,-\frac{1}{2}\right) \text{ and } \left(-\frac{1}{2},\infty\right)$$

Since the numerator of the derivative is the constant (-4) the numerator will always be negative. The denominator is squared so it will always be positive. Therefore, the derivative will always be negative, which means the function is always decreasing.



Intervals where f(x) is increasing: never

Intervals where f(x) is decreasing: $\left(-\infty,-\frac{1}{2}\right)$ and $\left(-\frac{1}{2},\infty\right)$

Step 7: Locate any relative extrema

Since the function has no critical numbers and is always decreasing, there are no relative extrema.

Step 8: Find the 2^{nd} derivative.

Since the numerator of the 1^{st} derivative is a constant it would be easier to find the 2^{nd} derivative by first rewriting the 1^{st} derivative in the form of a power function

$$f'(x)=\frac{-4}{(1+2x)^2}$$

$$f'(x)=-4(1+2x)^{-2}$$

Example 6 (Continued):

Step 8: Find the 2nd derivative.

Now use the generalized power rule to find the 2nd derivative.

$$f'(x) = -4(1+2x)^{-2}$$
$$f''(x) = -4(-2)(1+2x)^{-2-1} D_x(1+2x)$$
$$= 8(1+2x)^{-3}(2)$$
$$= \frac{16}{(1+2x)^3}$$

Step 9: Find the critical numbers for the 2nd derivative.

The numerator of the 2nd derivative is a constant number so the derivative will never be equal to zero, so the only possible critical numbers will come from the denominator.

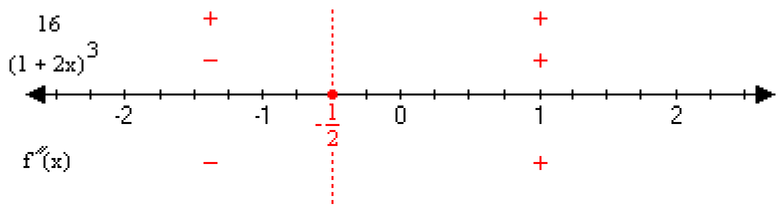
$$(1+2x)^3 = 0$$
$$1+2x = 0$$
$$2x = -1$$
$$x = -\frac{1}{2}$$

The only number that will make the denominator zero is –1/2. However, this cannot be a critical number since –1/2 is excluded from the domain of f(x).

Example 6 (Continued):

Step 10: Determine the intervals where the function is concave up or concave down.

The intervals that we must test are $\left(-\infty, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \infty\right)$.



The function is concave down on the interval $\left(-\infty, -\frac{1}{2}\right)$ and concave up on the interval $\left(-\frac{1}{2}, \infty\right)$

Step 11: Locate any points of inflection.

The only change in concavity occurs at x = -1/2 but this cannot be a point of inflection since it is not included in the domain of f(x).

Step 12: Sketch the graph

Now you will plot all of the points and asymptotes that you have found in the previous steps and then connect the points with a smooth curve. From steps 6 & 10 we know where the function is increasing/decreasing and its concavity.

For the interval $\left(-\infty, -\frac{1}{2}\right)$ the function is decreasing and concave down so the graph in this interval will look like 

For the interval $\left(-\frac{1}{2}, \infty\right)$ the function is decreasing and concave up so the graph in this interval will look like 

Example 6 (Continued):