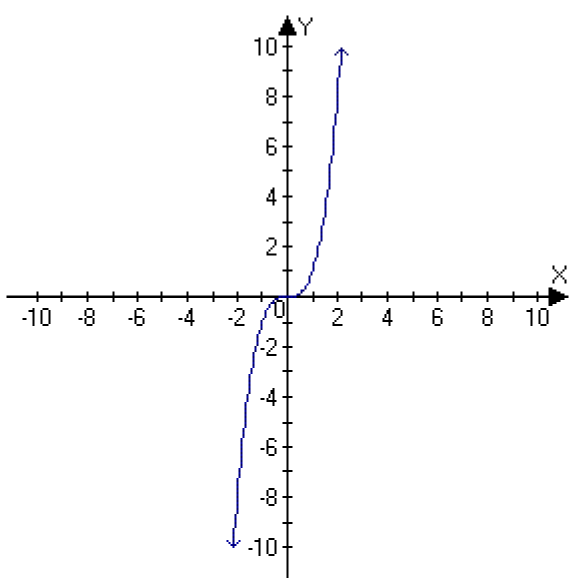
Optimization I

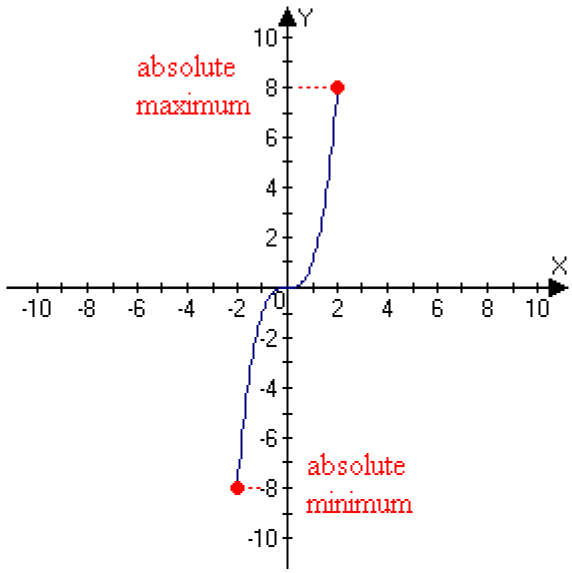
Absolute Extrema

The absolute extrema of a function are the largest and smallest possible y-value of a function. In the previous sections, you have seen where a function can have multiple relative maximums or minimums but when dealing with absolute extrema a function can never have more than one absolute maximum or absolute minimum.

It is also possible for a function to have neither an absolute maximum nor an absolute minimum if you are looking at the function over an open interval such as $(-\infty, \infty)$. Take the cubic function, f(x) = x^3, for example. On the open interval $(-\infty, \infty)$ the cubic function will have no absolute extrema because the y-values will continue to increase without bound as x increases and will continue to decrease without bound as x decreases.



Therefore, to ensure that we can find an absolute maximum and minimum for a function we will examine the function over a closed interval such as [a, b]. If we use the same cubic function as above but limit it to the interval [-2, 2] we can see that the absolute maximum is 8 and the absolute minimum is –8.



<u>Absolute Maximum or Minimum</u>

If we let f(x) be a function defined on the closed interval [a, b] and c be a number within this interval, then:

f(c) is the absolute maximum of the function on the interval if for all x-values in the interval $f(x) \leq f(c)$

f(c) is the absolute minimum of the function on the interval if for all x-values in the interval $f(x) \geq f(c)$

The number "c" can represent either the endpoints or any of the relative extrema for the function that are within the given interval. Therefore, when trying to find the absolute extrema of a function you will first want to find the critical number of the first derivative (which are the relative extrema).

Steps to follow in order to determine the absolute extrema of a function:

If the function is continuous on a closed interval [a, b] then

1. Find the first derivative of the function
2. Find all critical numbers of the first derivative that are within the open interval (a, b)
3. Evaluate the function at all critical numbers in the open interval (a, b)
4. Evaluate the function at the endpoints “a” and “b” of the closed interval [a, b]
5. Compare the y-values determined in steps 3 and 4.
    a. The largest value is the absolute maximum
    b. The smallest value is the absolute minimum

Example 1: Find the absolute extrema of the function $f(x) = x^3 + 2x^2 – 15x + 3$ on the interval [-4, 2].

Solution:

Step 1: Find the first derivative

$$
\begin{aligned}
f(x) &= x^3 + 2x^2 - 15x + 3 \\
f'(x) &= 3x^{3-1} + 2\left(2x^{2-1}\right) - 15x^{1-1} + 0 \\
&= 3x^2 + 4x - 15
\end{aligned}
$$

Step 2: Find the critical numbers within the interval (-4, 2)

$$
\begin{aligned}
f'(x) &= 3x^2 + 4x - 15 \\
0 &= 3x^2 + 4x - 15 \\
0 &= (3x-5)(x+3)
\end{aligned}
$$

$$
\begin{aligned}
3x - 5 &= 0 & x + 3 &= 0 \\
3x &= 5 & x &= -3 \\
x &= \frac{5}{3} & &
\end{aligned}
$$

Example 1 (Continued):

Step 3: Evaluate the function at the critical numbers from step 2

$$f(x) = x^3 + 2x^2 - 15x + 3$$
$$f\left(\frac{5}{3}\right) = \left(\frac{5}{3}\right)^3 + 2\left(\frac{5}{3}\right)^2 - 15\left(\frac{5}{3}\right) + 3$$
$$= \left(\frac{125}{27}\right) + 2\left(\frac{25}{9}\right) - \left(\frac{75}{3}\right) + 3$$
$$= \frac{125}{27} + \frac{50}{9} - 25 + 3$$
$$= \frac{125}{27} + \frac{150}{27} - 22$$
$$= \frac{275}{27} - \frac{594}{27}$$
$$= -\frac{319}{27}$$
$$\approx -11.8$$

$$f(x) = x^3 + 2x^2 - 15x + 3$$
$$f(-3) = (-3)^3 + 2(-3)^2 - 15(-3) + 3$$
$$= (-27) + 2(9) + 45 + 3$$
$$= -27 + 18 + 48$$
$$= 39$$

Step 4: Evaluate the function at the interval endpoints

$$f(x) = x^3 + 2x^2 - 15x + 3$$
$$f(-4) = (-4)^3 + 2(-4)^2 - 15(-4) + 3$$
$$= (-64) + 2(16) + 60 + 3$$
$$= -64 + 32 + 63$$
$$= 31$$

$$f(x) = x^3 + 2x^2 - 15x + 3$$
$$f(2) = (2)^3 + 2(2)^2 - 15(2) + 3$$
$$= (8) + 2(4) - 30 + 3$$
$$= 8 + 8 - 27$$
$$= -11$$

Example 1 (Continued):

Step 5: Compare the y-values

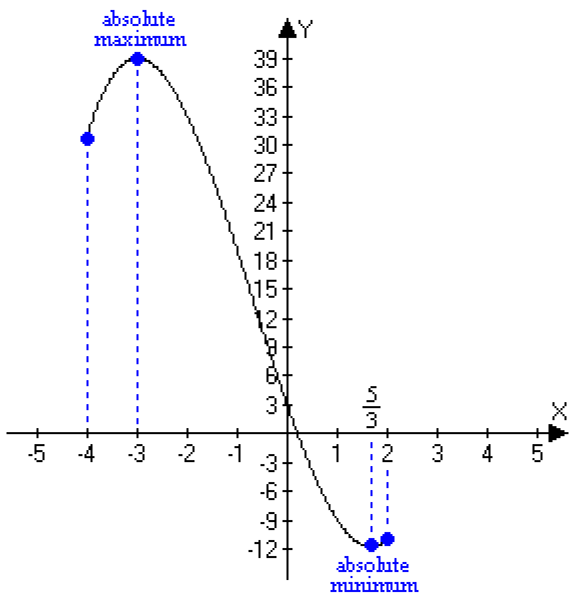
$$f(-4) = 31$$

$$f(-3) = 39$$

$$f\left(\frac{5}{3}\right) = -11.8$$

$$f(2) = -11$$

The largest value is 39 and the smallest value is –11.8 so these are the absolute maximum and absolute minimum values for the function in the interval [-4, 2] as we can see in the graph of the function.



Example 2: Find the absolute extrema (if any exist) for the function $f(x)=\frac{x}{x^2+1}$.

Solution:

Step 1: Find the first derivative

$$f(x)=\frac{x}{x^2+1}$$

$$f'(x)=\frac{\left(x^2+1\right)D_x(x)-(x)D_x\left(x^2+1\right)}{\left(x^2+1\right)^2}$$

$$=\frac{\left(x^2+1\right)(1)-(x)(2x)}{\left(x^2+1\right)^2}$$

$$=\frac{x^2+1-2x^2}{\left(x^2+1\right)^2}$$

$$=\frac{-x^2+1}{\left(x^2+1\right)^2}$$

Step 2: Find the critical numbers

$$f'(x)=\frac{-x^2+1}{\left(x^2+1\right)^2}$$

Since the derivative is a fraction you would find the critical numbers by setting the numerator and denominator equal to zero and solve for x.

$-x^2+1=0$	$\left(x^2+1\right)^2=0$
$-x^2=-1$	$x^2+1=0$
$x^2=1$	$x^2=-1$
$x=\sqrt{1}$	$x=\sqrt{-1}$
$x=\pm 1$	*not a real number*

Example 2 (Continued):

Step 3: Evaluate the function at the critical numbers from step 2

$$f(x) = \frac{x}{x^2+1} \qquad\qquad f(x) = \frac{x}{x^2+1}$$

$$f(-1) = \frac{-1}{(-1)^2+1} \qquad\qquad f(1) = \frac{1}{(1)^2+1}$$

$$= \frac{-1}{1+1} \qquad\qquad = \frac{1}{1+1}$$

$$= -\frac{1}{2} \qquad\qquad = \frac{1}{2}$$

Step 4: Evaluate the function at the interval endpoints

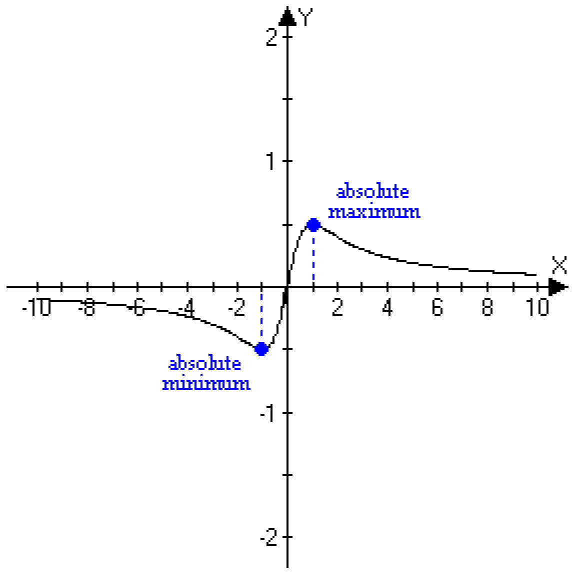
In this problem we were not given an interval. However, if we look at the function as x approaches both positive and negative infinity we can see that the function approaches 0.

Step 5: Compare the y-values

$$f(-1) = -\frac{1}{2}$$
$$f(1) = \frac{1}{2}$$

Since these are the only the critical points for the function and the function approaches 0 on both ends the absolute maximum is ½ and the absolute minimum is – ½ , which can be see better in the graph of the function.

Example 2 (Continued):



Example 3: Find the absolute extrema of the function f(x) = (x + 1)(x + 2)^2 on the interval [-4, 0].

Solution:

Step 1: Find the first derivative

$$f(x) = (x+1)(x+2)^2$$

$$f'(x) = (x+1)D_x(x+2)^2 + (x+2)^2 D_x(x+1)$$

$$= (x+1)(2)(x+2)(1) + (x+2)^2(1)$$

$$= (2x+2)(x+2) + (x+2)^2$$

$$= (x+2)\left[(2x+2) + (x+2)\right]$$

$$= (x+2)(3x+4)$$

Example 3 (Continued):

Step 2: Find the critical numbers contained in the interval (-4, 0)

$$\begin{aligned} 3x+4&=0 \\ 3x&=-4 \\ x&=-\frac{4}{3} \end{aligned} \qquad \begin{aligned} x+2&=0 \\ x&=-2 \end{aligned}$$

Both of these numbers are within the interval (-4, 0)

Step 3: Evaluate the function at the critical numbers from step 2

$$\begin{aligned} f(x)&=(x+1)(x+2)^2 \\ f(-2)&=(-2+1)(-2+2)^2 \\ &=(-1)(0)^2 \\ &=0 \end{aligned}$$

$$\begin{aligned} f(x)&=(x+1)(x+2)^2 \\ f\left(-\frac{4}{3}\right)&=\left(-\frac{4}{3}+1\right)\left(-\frac{4}{3}+2\right)^2 \\ &=\left(-\frac{4}{3}+\frac{3}{3}\right)\left(-\frac{4}{3}+\frac{6}{3}\right)^2 \\ &=\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 \\ &=\left(-\frac{1}{3}\right)\left(\frac{4}{9}\right) \\ &=-\frac{4}{27} \end{aligned}$$

Example 3 (Continued):

Step 4: Evaluate the function at the interval endpoints –4 and 0

$$\begin{aligned} f(x) &= (x+1)(x+2)^2 \\ f(-4) &= (-4+1)(-4+2)^2 \\ &= (-3)(-2)^2 \\ &= (-3)(4) \\ &= -12 \end{aligned}$$

$$\begin{aligned} f(x) &= (x+1)(x+2)^2 \\ f(0) &= (0+1)(0+2)^2 \\ &= (1)(2)^2 \\ &= (1)(4) \\ &= 4 \end{aligned}$$

Step 5: Compare the y-values

$$f(-4) = -12 \qquad f\left(-\frac{4}{3}\right) = -\frac{4}{27}$$

$$f(-2) = 0 \qquad f(0) = 4$$

The absolute maximum is 4 and the absolute minimum is –12.