Review Exercise Set 13

Exercise 1: Find the intervals where the function in the given graph is concave upward or concave downward, and any points of inflection.

Concave up:
Concave down:
Point of inflection:

Exercise 2: Find the intervals where the given function is concave upward or concave downward, and any points of inflection.

\[ f(x) = x^4 - 4x^3 + 10 \]
Exercise 3: Find the intervals where the given function is concave upward or concave downward, and any points of inflection.

\[ f(x) = -x^3 - 12x^2 - 45x + 10 \]

Exercise 4: Find the relative extrema, if any, of the given function using the second derivative test, if applicable.

\[ y = (x + 1)^2 / (1 + x^2) \]
Exercise 5: Find the relative extrema, if any, of the given function using the second derivative test, if applicable.

\[ f(x) = 2x^3 - 4x^2 + 2 \]
Review Exercise Set 13 Answer Key

Exercise 1: Find the intervals where the function in the given graph is concave upward or concave downward, and any points of inflection.

Concave up: \((-\infty, 1.33)\)

Concave down: \((1.33, \infty)\)

Point of inflection: \((1.33, 2.73)\)

Exercise 2: Find the intervals where the given function is concave upward or concave downward, and any points of inflection.

\[ f(x) = x^4 - 4x^3 + 10 \]

Find the first and second derivatives

\[ f'(x) = 4x^3 - 12x^2 \]
\[ f''(x) = 12x^2 - 24x \]

Find the critical numbers of the second derivative

\[ 0 = 12x^2 - 24x \]
\[ 0 = 12x(x - 2) \]
\[ 12x = 0 \text{ or } x - 2 = 0 \]
\[ x = 0 \text{ or } x = 2 \]
Exercise 2 (Continued):

Test each interval to determine the sign of the second derivative

\[ f''(x) = 12x(x - 2); \ x = -1 \]
\[ f''(-1) = 12(-1)(-1 - 2) \]
\[ f''(-1) = (-12)(-3) \]
\[ f''(-1) = 36 \]

\[ f''(x) = 12x(x - 2); \ x = 1 \]
\[ f''(1) = 12(1)(1 - 2) \]
\[ f''(1) = (12)(-1) \]
\[ f''(1) = -12 \]

\[ f''(x) = 12x(x - 2); \ x = 3 \]
\[ f''(3) = 12(3)(3 - 2) \]
\[ f''(3) = (36)(1) \]
\[ f''(3) = 36 \]

Concave up: \((-\infty, 0) \cup (2, \infty)\)
Concave down: \((0, 2)\)

Since the concavity changes at \(x = 0\) and \(x = 2\), these would be the locations of the points of inflection.

Evaluate \(f(x)\) at 0 and 2 to find the points of inflection

\[ f(x) = x^4 - 4x^3 + 10; \ x = 0 \]
\[ f(0) = (0)^4 - 4(0)^3 + 10 \]
\[ f(0) = 10 \]

\[ f(x) = x^4 - 4x^3 + 10; \ x = 2 \]
\[ f(2) = (2)^4 - 4(2)^3 + 10 \]
\[ f(2) = 16 - 32 + 10 \]
\[ f(2) = -6 \]

The points of inflection are at \((0, 10)\) and \((2, -6)\).

Exercise 3: Find the intervals where the given function is concave upward or concave downward, and any points of inflection.

\[ f(x) = -x^3 - 12x^2 - 45x + 10 \]

Find the first and second derivatives

\[ f'(x) = -3x^2 - 24x - 45 \]
\[ f''(x) = -6x - 24 \]
Exercise 3 (Continued):

Find the critical numbers of the second derivative

\[ 0 = -6x - 24 \]
\[ 6x = -24 \]
\[ x = -4 \]

Test each interval to determine the sign of the second derivative

\[ f''(x) = -6x - 24; x = -5 \]
\[ f''(-5) = -6(-5) - 24 \]
\[ f''(-5) = 30 - 24 \]
\[ f''(-5) = 6 \]

\[ f''(x) = -6x - 24; x = 0 \]
\[ f''(0) = -6(0) - 24 \]
\[ f''(0) = -24 \]

Concave up: \((-\infty, -4)\)
Concave down: \((-4, \infty)\)

Since the concavity changes at the critical number \(x = -4\) this is the location of the point of inflection.

Evaluate \(f(x)\) at \(x = -4\) to find the point of inflection

\[ f(x) = -x^3 - 12x^2 - 45x + 10; x = -4 \]
\[ f(-4) = -(-4)^3 - 12(-4)^2 - 45(-4) + 10 \]
\[ f(-4) = 64 - 192 + 180 + 10 \]
\[ f(-4) = 62 \]

The point of inflection is at \((-4, 62)\).

Exercise 4: Find the relative extrema, if any, of the given function using the second derivative test, if applicable.

\[ y = (x + 1)^2 / (1 + x^2) \]

Find the first derivative

\[ D_x(x + 1)^2 = 2(x + 1)(1) = 2(x + 1) \]
\[ D_x(1 + x^2) = 2x \]
Exercise 4 (Continued):

\[
y' = \frac{(1 + x^2)D_x(x+1)^2 - (x+1)^2 D_x(1 + x^2)}{(1 + x^2)^2}
\]

\[
= \frac{(1 + x^2)2(x+1) - (x+1)^2 (2x)}{(1 + x^2)^2}
\]

\[
= \frac{2(x+1)[(1 + x^2) - (x+1)(x)]}{(1 + x^2)^2}
\]

\[
= \frac{2(x+1)(1-x)}{(1 + x^2)^2}
\]

Find the critical numbers of the first derivative

Set each factor in the numerator and denominator equal to zero and solve for x

\[
x + 1 = 0 \quad 1 - x = 0 \quad 1 + x^2 = 0
\]

\[
x = -1 \quad x = 1 \quad x^2 = -1
\]

\[x^2\] cannot be a negative number, so there are no critical numbers from this factor.

The critical numbers of the first derivative are -1 and 1

Find the second derivative

\[
y'' = \frac{2 - 2x^2}{(1 + x^2)^2}
\]

\[D_x(2 - 2x^2) = -4x\]

\[D_x(1 + x^2)^2 = 2(1 + x^2)(2x) = 4x(1 + x^2)\]
Exercise 4 (Continued):

\[
y'' = \frac{(1+x^2)^2 D_x \left( 2 - 2x^2 \right) - \left( 2 - 2x^2 \right) D_x \left( 1 + x^2 \right)^2}{\left[ (1+x^2)^2 \right]^2}
\]

\[
= \frac{(1+x^2)^2 (-4x) - \left( 2 - 2x^2 \right)(4x)(1+x^2)}{(1+x^2)^4}
\]

\[
= 4x(1+x^2)\left[ (1+x^2)(-1) - (2-2x^2) \right]
\]

\[
= 4x \left( 3 + x^2 \right)
\]

Evaluate the second derivative at the first derivative critical numbers

\[
y'' = \frac{4x \left( 3 + x^2 \right)}{(1+x^2)^3}; \quad x = -1
\]

\[
y'' = \frac{4(-1)(-3+(-1)^2)}{(1+(-1)^2)^3}
\]

\[
= \frac{(-4)(-2)}{2^3}
\]

\[
= \frac{8}{8}
\]

\[
= 1
\]

Second derivative is positive so a relative minimum is at \( x = -1 \)

\[
y'' = \frac{4x \left(3 + x^2 \right)}{(1+x^2)^3}; \quad x = 1
\]
Exercise 4 (Continued):

\[ y'' = \frac{4(1)(-3 + (1)^2)}{(1 + (1)^2)^3} \]
\[ = \frac{(4)(-2)}{(2)^3} \]
\[ = -\frac{8}{8} \]
\[ = -1 \]

Second derivative is negative so a relative maximum is at \( x = 1 \)

Evaluate the original function at -1 and 1 to find the relative extrema

\[ y = \frac{(x + 1)^2}{1 + x^2}; \quad x = -1 \]
\[ y = \frac{[(-1) + 1]^2}{1 + (-1)^2} \]
\[ y = \frac{0}{2} \]
\[ y = 0 \]

relative minimum is at (-1, 0)

\[ y = \frac{(x + 1)^2}{1 + x^2}; \quad x = 1 \]
\[ y = \frac{[1 + 1]^2}{1 + 1^2} \]
\[ y = \frac{4}{2} \]
\[ y = 2 \]

relative maximum is at (1, 2)

Exercise 5: Find the relative extrema, if any, of the given function using the second derivative test, if applicable.

\[ f(x) = 2x^3 - 4x^2 + 2 \]

Find the first derivative

\[ f'(x) = 6x^2 - 8x \]
Exercise 5 (Continued):

Find the critical numbers of the first derivative

\[ 0 = 6x^2 - 8x \]
\[ 0 = 2x(3x - 4) \]
\[ 2x = 0 \text{ or } 3x - 4 = 0 \]
\[ x = 0 \text{ or } x = \frac{4}{3} \]

Find the second derivative

\[ f''(x) = 12x - 8 \]

Evaluate the second derivative at the first derivative critical numbers

\[ f''(x) = 12x - 8; \ x = 0 \]
\[ f''(0) = 12(0) - 8 \]
\[ f''(0) = -8 \]

Second derivative is negative so a relative maximum is at \( x = 0 \)

\[ f''(x) = 12x - 8; \ x = \frac{4}{3} \]
\[ = 12\left(\frac{4}{3}\right) - 8 \]
\[ = 16 - 8 \]
\[ = 8 \]

Second derivative is positive so a relative minimum is at \( x = \frac{4}{3} \)

Evaluate the original function at 0 and \( \frac{4}{3} \) to find the relative extrema

\[ f(x) = 2x^3 - 4x^2 + 2; \ x = 0 \]
\[ f(0) = 2(0)^3 - 4(0)^2 + 2 \]
\[ f(0) = 2 \]

relative maximum is at \( (0, 2) \)
Exercise 5 (Continued):

\[ f(x) = 2x^3 - 4x^2 + 2; \quad x = \frac{4}{3} \]

\[ f\left( \frac{4}{3} \right) = 2\left( \frac{4}{3} \right)^3 - 4\left( \frac{4}{3} \right)^2 + 2 \]

\[ = 2\left( \frac{64}{27} \right) - 4\left( \frac{16}{9} \right) + 2 \]

\[ = \frac{128}{27} - \frac{192}{27} + \frac{54}{27} \]

\[ = \frac{-10}{27} \]

Relative maximum is at \( \left( \frac{4}{3}, \frac{-10}{27} \right) \).