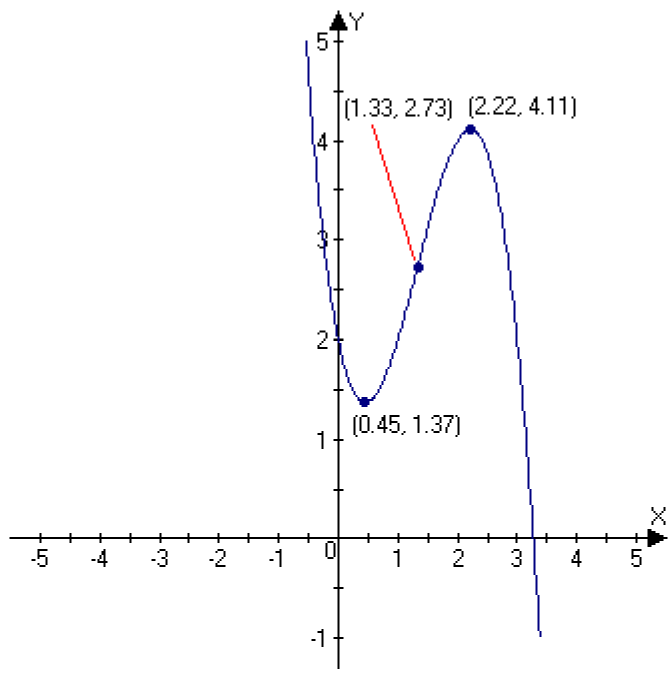
Review Exercise Set 13

Exercise 1: Find the intervals where the function in the given graph is concave upward or concave downward, and any points of inflection.



Concave up:

Concave down:

Point of inflection:

Exercise 2: Find the intervals where the given function is concave upward or concave downward, and any points of inflection.

$$f(x) = x^4 - 4x^3 + 10$$

Exercise 3: Find the intervals where the given function is concave upward or concave downward, and any points of inflection.

$$f(x) = -x^3 - 12x^2 - 45x + 10$$

Exercise 4: Find the relative extrema, if any, of the given function using the second derivative test, if applicable.

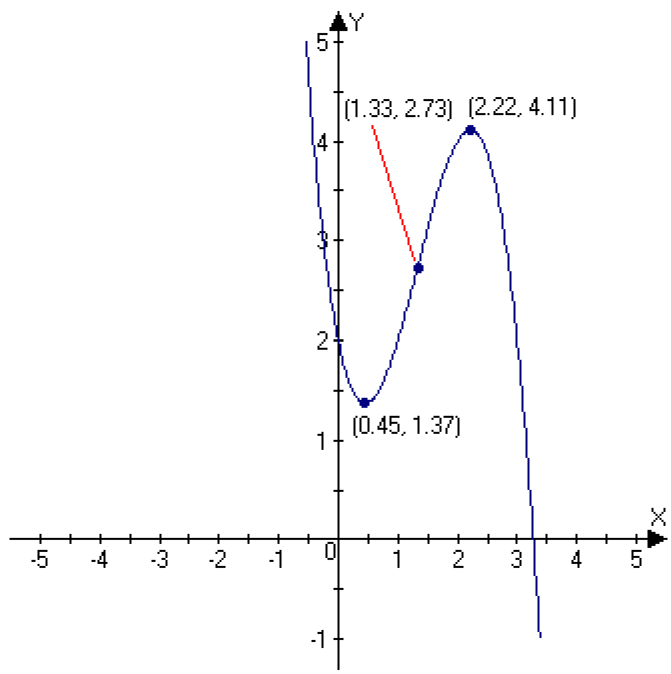
$$y = (x + 1)^2 / (1 + x^2)$$

Exercise 5: Find the relative extrema, if any, of the given function using the second derivative test, if applicable.

$$f(x) = 2x^3 - 4x^2 + 2$$

Review Exercise Set 13 Answer Key

Exercise 1: Find the intervals where the function in the given graph is concave upward or concave downward, and any points of inflection.



Concave up: $(-\infty, 1.33)$

Concave down: $(1.33, \infty)$

Point of inflection: (1.33, 2.73)

Exercise 2: Find the intervals where the given function is concave upward or concave downward, and any points of inflection.

$f(x) = x^4 - 4x^3 + 10$

Find the first and second derivatives

$f'(x) = 4x^3 - 12x^2$
$f''(x) = 12x^2 - 24x$

Find the critical numbers of the second derivative

$0 = 12x^2 - 24x$
$0 = 12x(x - 2)$
$12x = 0$ or $x - 2 = 0$
$x = 0$ or $x = 2$

Exercise 2 (Continued):

Test each interval to determine the sign of the second derivative

$f''(x) = 12x(x - 2); x = -1$
$f''(-1) = 12(-1)(-1 - 2)$
$f''(-1) = (-12)(-3)$
$f''(-1) = 36$

$f''(x) = 12x(x - 2); x = 1$
$f''(1) = 12(1)(1 - 2)$
$f''(1) = (12)(-1)$
$f''(1) = -12$

$f''(x) = 12x(x - 2); x = 3$
$f''(3) = 12(3)(3 - 2)$
$f''(3) = (36)(1)$
$f''(3) = 36$

Concave up: $(-\infty, 0) \cup (2, \infty)$
Concave down: (0, 2)

Since the concavity changes at x = 0 and x = 2, these would be the locations of the points of inflection.

Evaluate f(x) at 0 and 2 to find the points of inflection

$f(x) = x^4 - 4x^3 + 10; x = 0$
$f(0) = (0)^4 - 4(0)^3 + 10$
$f(0) = 10$

$f(x) = x^4 - 4x^3 + 10; x = 2$
$f(2) = (2)^4 - 4(2)^3 + 10$
$f(2) = 16 - 32 + 10$
$f(2) = -6$

The points of inflection are at (0, 10) and (2, -6).

Exercise 3: Find the intervals where the given function is concave upward or concave downward, and any points of inflection.

$f(x) = -x^3 - 12x^2 - 45x + 10$

Find the first and second derivatives

$f'(x) = -3x^2 - 24x - 45$
$f''(x) = -6x - 24$

Exercise 3 (Continued):

Find the critical numbers of the second derivative

$0 = -6x - 24$
$6x = -24$
$x = -4$

Test each interval to determine the sign of the second derivative

$f''(x) = -6x - 24;\ x = -5$
$f''(-5) = -6(-5) - 24$
$f''(-5) = 30 - 24$
$f''(-5) = 6$

$f''(x) = -6x - 24;\ x = 0$
$f''(0) = -6(0) - 24$
$f''(0) = -24$

Concave up: $(-\infty, -4)$
Concave down: $(-4, \infty)$

Since the concavity changes at the critical number x = -4 this is the location of the point of inflection.

Evaluate f(x) at x = -4 to find the point of inflection

$f(x) = -x^3 - 12x^2 - 45x + 10;\ x = -4$
$f(-4) = -(-4)^3 - 12(-4)^2 - 45(-4) + 10$
$f(-4) = 64 - 192 + 180 + 10$
$f(-4) = 62$

The point of inflection is at (-4, 62).

Exercise 4: Find the relative extrema, if any, of the given function using the second derivative test, if applicable.

$y = (x + 1)^2 / (1 + x^2)$

Find the first derivative

$D_x(x + 1)^2 = 2(x + 1)(1) = 2(x + 1)$

$D_x(1 + x^2) = 2x$

Exercise 4 (Continued):

$$y' = \frac{\left(1+x^2\right)D_x\left(x+1\right)^2 - \left(x+1\right)^2 D_x\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$= \frac{\left(1+x^2\right)2\left(x+1\right) - \left(x+1\right)^2\left(2x\right)}{\left(1+x^2\right)^2}$$

$$= \frac{2\left(x+1\right)\left[\left(1+x^2\right) - \left(x+1\right)\left(x\right)\right]}{\left(1+x^2\right)^2}$$

$$= \frac{2\left(x+1\right)\left(1-x\right)}{\left(1+x^2\right)^2}$$

Find the critical numbers of the first derivative

Set each factor in the numerator and denominator equal to zero and solve for x

x + 1 = 0	1 - x = 0	$1 + x^2 = 0$
x = -1	x = 1	$x^2 = -1$

x^2 cannot be a negative number, so there are no critical numbers from this factor.

The critical numbers of the first derivative are -1 and 1

Find the second derivative

$$y' = \frac{2-2x^2}{\left(1+x^2\right)^2}$$

$D_x(2 - 2x^2) = -4x$

$D_x(1 + x^2)^2 = 2(1 + x^2)(2x) = 4x(1 + x^2)$

Exercise 4 (Continued):

$$y'' = \frac{\left(1+x^2\right)^2 D_x\left(2-2x^2\right)-\left(2-2x^2\right)D_x\left(1+x^2\right)^2}{\left[\left(1+x^2\right)^2\right]^2}$$

$$= \frac{\left(1+x^2\right)^2(-4x)-\left(2-2x^2\right)(4x)\left(1+x^2\right)}{\left(1+x^2\right)^4}$$

$$= \frac{4x\left(1+x^2\right)\left[\left(1+x^2\right)(-1)-\left(2-2x^2\right)\right]}{\left(1+x^2\right)^4}$$

$$= \frac{4x\left(-3+x^2\right)}{\left(1+x^2\right)^3}$$

Evaluate the second derivative at the first derivative critical numbers

$$y'' = \frac{4x\left(-3+x^2\right)}{\left(1+x^2\right)^3}; \text{ x = -1}$$

$$y'' = \frac{4(-1)\left(-3+(-1)^2\right)}{\left(1+(-1)^2\right)^3}$$

$$= \frac{(-4)(-2)}{(2)^3}$$

$$= \frac{8}{8}$$

$$= 1$$

Second derivative is positive so a relative minimum is at x = -1

$$y'' = \frac{4x\left(-3+x^2\right)}{\left(1+x^2\right)^3}; \text{ x = 1}$$

Exercise 4 (Continued):

$$y'' = \frac{4(1)\left(-3+(1)^2\right)}{\left(1+(1)^2\right)^3}$$
$$= \frac{(4)(-2)}{(2)^3}$$
$$= \frac{-8}{8}$$
$$= -1$$

Second derivative is negative so a relative maximum is at x = 1

Evaluate the original function at -1 and 1 to find the relative extrema

$y = (x + 1)^2 / (1 + x^2)$; $x = -1$

$$y = \frac{\left[(-1)+1\right]^2}{1+(-1)^2}$$

$$y = \frac{0}{2}$$

y = 0

relative minimum is at (-1, 0)

$y = (x + 1)^2 / (1 + x^2)$; $x = 1$

$$y = \frac{\left[1+1\right]^2}{1+1^2}$$

$$y = \frac{4}{2}$$

y = 2

relative maximum is at (1, 2)

Exercise 5: Find the relative extrema, if any, of the given function using the second derivative test, if applicable.

$f(x) = 2x^3 - 4x^2 + 2$

Find the first derivative

$f'(x) = 6x^2 - 8x$

Exercise 5 (Continued):

Find the critical numbers of the first derivative

$0 = 6x^2 - 8x$
$0 = 2x(3x - 4)$
$2x = 0$ or $3x - 4 = 0$
$x = 0$ or $x = \frac{4}{3}$

Find the second derivative

$f''(x) = 12x - 8$

Evaluate the second derivative at the first derivative critical numbers

$f''(x) = 12x - 8; x = 0$
$f''(0) = 12(0) - 8$
$f''(0) = -8$

Second derivative is negative so a relative maximum is at $x = 0$

$f''(x) = 12x - 8; x = \frac{4}{3}$
$= 12(\frac{4}{3}) - 8$
$= 16 - 8$
$= 8$

Second derivative is positive so a relative minimum is at $x = \frac{4}{3}$

Evaluate the original function at 0 and $\frac{4}{3}$ to find the relative extrema

$f(x) = 2x^3 - 4x^2 + 2; x = 0$
$f(0) = 2(0)^3 - 4(0)^2 + 2$
$f(0) = 2$

relative maximum is at (0, 2)

Exercise 5 (Continued):

$$f(x) = 2x^3 - 4x^2 + 2; \; x = \frac{4}{3}$$

$$f(\frac{4}{3}) = 2(\frac{4}{3})^3 - 4(\frac{4}{3})^2 + 2$$

$$= 2(\frac{64}{27}) - 4(\frac{16}{9}) + 2$$

$$= \frac{128}{27} - \frac{192}{27} + \frac{54}{27}$$

$$= -\frac{10}{27}$$

relative maximum is at $(\frac{4}{3}, -\frac{10}{27})$