Review Exercise Set 15

Exercise 1: Indicate the location of the absolute extrema (if any) for the function in the given graph.

Absolute minimum:

Absolute maximum:

Exercise 2: Find all absolute extrema within the specified closed interval for the function:

$$g(x) = x^3 - x^2 + 8 ; [-1, 1]$$
Exercise 3: Find all absolute extrema within the specified closed interval for the function:

\[ y = \frac{3}{x^2 + 2} ; \quad [0, 4] \]

Exercise 4: Find all absolute extrema (if they exist) for the function:

\[ f(x) = x^4 - 4x^3 + 4x^2 + 4 \]
Exercise 5: If a manufacturing company's cost function is defined as \( c(x) = 2000 + 96x + 4x^{3/2} \) (where \( x \) represents thousands of units), at what production level would the average cost be minimized?
Exercise 1: Indicate the location of the absolute extrema (if any) for the function in the given graph.

- Absolute minimum: \( x_1 \) and \( x_3 \)
- Absolute maximum: none, because the function increasing without bound as \( x \) approaches positive or negative infinity. \( x_2 \) is a relative maximum but not an absolute maximum.

Exercise 2: Find all absolute extrema within the specified closed interval for the function:

\[ g(x) = x^3 - x^2 + 8 \; ; \; [-1, 1] \]

Find the first derivative

\[ g'(x) = 3x^2 - 2x \]

Find the critical numbers of the first derivative

\[
\begin{align*}
0 &= 3x^2 - 2x \\
0 &= x(3x - 2) \\
x &= 0 \text{ or } 3x - 2 = 0 \\
x &= 0 \text{ or } x = \frac{2}{3}
\end{align*}
\]
Exercise 2 (Continued):

Evaluate the function at the critical numbers and interval endpoints

\[ g(x) = x^3 - x^2 + 8; \quad x = -1 \]
\[ g(-1) = (-1)^3 - (-1)^2 + 8 \]
\[ g(-1) = -1 - 1 + 8 \]
\[ g(-1) = 6 \]

\[ g(x) = x^3 - x^2 + 8; \quad x = 0 \]
\[ g(0) = (0)^3 - (0)^2 + 8 \]
\[ g(0) = 0 - 0 + 8 \]
\[ g(0) = 8 \]

\[ g(x) = x^3 - x^2 + 8; \quad x = \frac{2}{3} \]
\[ g\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^2 + 8 \]
\[ g\left(\frac{2}{3}\right) = \frac{8}{27} - \frac{4}{9} + 8 \]
\[ g\left(\frac{2}{3}\right) = \frac{212}{27} \]
\[ g\left(\frac{2}{3}\right) = \frac{7}{27} \]

\[ g(x) = x^3 - x^2 + 8; \quad x = 1 \]
\[ g(1) = (1)^3 - (1)^2 + 8 \]
\[ g(1) = 1 - 1 + 8 \]
\[ g(1) = 8 \]

Identify the absolute extrema

Absolute minimum: (-1, 6)

Absolute maximum: (0, 8) and (1, 8)

Exercise 3: Find all absolute extrema within the specified closed interval for the function:

\[ y = \frac{3}{x^2 + 2}; \quad [0, 4] \]

Find the first derivative

\[ y = 3(x^2 + 2)^{-1} \]
\[ y' = -3(x^2 + 2)^{-2}(2x) \]
\[ y' = \frac{-6x}{(x^2 + 2)^2} \]
Exercise 3 (Continued):

Find the critical numbers of the first derivative

Set numerator equal to zero

\[-6x = 0\]
\[x = 0\]

Set denominator equal to zero

\[(x^2 + 2)^2 = 0\]
\[x^2 + 2 = 0\]
\[x^2 = -2\]

\[x^2\] cannot be negative, so the denominator produces no critical numbers

Evaluate the function at the critical numbers and interval endpoints

\[y = \frac{3}{x^2 + 2}; x = 0\]
\[y = \frac{3}{x^2 + 2}; x = 4\]
\[y = \frac{3}{(0)^2 + 2} = \frac{3}{2}\]
\[y = \frac{3}{(4)^2 + 2} = \frac{3}{18}\]

Identify the absolute extrema

Absolute minimum: \((4, \frac{3}{18})\)

Absolute maximum: \((0, \frac{3}{2})\)

Exercise 4: Find all absolute extrema (if they exist) for the function:

\[f(x) = x^4 - 4x^3 + 4x^2 + 4\]

Find the first derivative

\[f'(x) = 4x^3 - 12x^2 + 8x\]
Exercise 4 (Continued):

Find the critical numbers of the first derivative

\[ 0 = 4x(x^2 - 3x + 2) \]
\[ 0 = 4x(x - 1)(x - 2) \]
\[ 4x = 0 \text{ or } x - 1 = 0 \text{ or } x - 2 = 0 \]
\[ x = 0 \text{ or } x = 1 \text{ or } x = 2 \]

Evaluate the function at the critical numbers

\[ f(x) = x^4 - 4x^3 + 4x^2 + 4; \quad x = 0 \]
\[ f(0) = (0)^4 - 4(0)^3 + 4(0)^2 + 4 \]
\[ f(0) = 4 \]

\[ f(x) = x^4 - 4x^3 + 4x^2 + 4; \quad x = 1 \]
\[ f(1) = (1)^4 - 4(1)^3 + 4(1)^2 + 4 \]
\[ f(1) = 1 - 4 + 4 + 4 \]
\[ f(1) = 5 \]

\[ f(x) = x^4 - 4x^3 + 4x^2 + 4; \quad x = 2 \]
\[ f(2) = (2)^4 - 4(2)^3 + 4(2)^2 + 4 \]
\[ f(2) = 16 - 32 + 16 + 4 \]
\[ f(2) = 4 \]

Identify the absolute extrema

Absolute minimum: (0, 4) and (2, 4)

Absolute maximum: none, because this is a function that increases without bound as \( x \) approaches positive and negative infinity.
Exercise 5: If a manufacturing company's cost function is defined as \( c(x) = 2000 + 96x + 4x^{3/2} \) (where \( x \) represents thousands of units), at what production level would the average cost be minimized?

Find the average cost function

\[
\overline{C}(x) = \frac{C(x)}{x} = \frac{2000 + 96x + 4x^{3/2}}{x} = \frac{2000}{x} + \frac{96x}{x} + \frac{4x^{3/2}}{x} = 2000x^{-1} + 96 + 4x^{1/2}
\]

Find the marginal average cost function

\[
\overline{C}'(x) = -2000x^{-2} + 2x^{-1/2}
\]

Find the critical numbers for the marginal average cost function

\[
0 = -2000x^2 + 2x^{3/2}
\]

\[
0 = x^2(-2000 + 2x^{3/2})
\]

\[
0 = -2000 + 2x^{3/2}
\]

\[
0 = \frac{2000}{x^2}
\]

Set numerator equal to zero

\[
0 = -2000 + 2x^{3/2}
\]

\[
2000 = 2x^{3/2}
\]

\[
1000 = x^{3/2}
\]

\[
1000^{2/3} = x
\]

\[
100 = x
\]

Set denominator equal to zero

\[
0 = x^2
\]

\[
0 = x
\]

The average cost function is not defined when \( x \) is zero so this cannot be a location for the absolute minimum.
Exercise 5 (Continued):

Evaluate the average cost function at the critical numbers

\[
\bar{C}(x) = \frac{2000}{x} + 96 + 4x^{1/2}
\]

\[
\bar{C}(100) = \frac{2000}{100} + 96 + 4(100)^{1/2}
\]

\[
= 20 + 96 + 40
\]

\[
= 156
\]

Identify the absolute minimum

Since \( x \) is in thousands of units, the average cost would be minimized when the production level is at 100,000 units.