The Chain Rule

The chain rule is used to find the derivative of a function that is the composition of two other functions. These composite functions are normally written as f[g(x)] or g[f(x)]. Breaking a composite function into simpler functions will make finding the derivative easier.

To review composite functions, lets say you are given the function $h(x) = (3x - 5)^4$. You can express the function h(x) as the composition of functions by letting g(x) equal the function inside the parenthesis, which in this case is 3x – 5, and f(x) equal the power function of x^4.

If $f(x) = x^4$ and g(x) = 3x – 5, then h(x) = f[g(x)]

$$\begin{aligned} h(x) &= f[g(x)] \\ &= f[3x - 5] \\ &= (3x - 5)^4 \end{aligned}$$

Example 1: Express the following function as the composition of two functions f and g, so that h(x) = f[g(x)].

$$h(x) = \sqrt[4]{\left(2x^2 - 7\right)^3}$$

Solution:

Step 1: Rewrite the radical using fractional exponents

$$\begin{aligned} h(x) &= \sqrt[4]{\left(2x^2 - 7\right)^3} \\ &= \left(2x^2 - 7\right)^{3/4} \end{aligned}$$

Step 2: Separate h(x) into the two functions f(x) and g(x)

$$f(x) = x^{3/4} \text{ and } g(x) = 2x^2 - 7$$

The derivative of the composite function can now be found by getting the derivatives of the two functions, f and g, and then substituting the functions into the chain rule formula. There are two ways in which the chain rule can be displayed. Both formulas are the same and will result in the same answer. The only difference is whether or not the substitution method is used.

Chain Rule (using substitution)

If y is a function of u and u is a function of x, then y can be expressed as a function of x.

$$y = f(u) \qquad u = g(x) \qquad y = f\left[g(x)\right]$$

The derivative can then be determined as:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 2: Find the derivative of the function $y = (3x-2)^{4/3}$.

Solution: For this example we will show how the substitution method would be used.

Step 1: Substitute u for the function g(x).

Let u = 3x – 2

$$y = (3x-2)^{4/3}$$
$$y = u^{4/3}$$

Step 2: Find the derivative of y and u

$$y = u^{4/3}$$
$$\frac{dy}{du} = \frac{4}{3}u^{4/3-1}$$
$$= \frac{4}{3}u^{4/3-3/3}$$
$$= \frac{4}{3}u^{1/3}$$

$$u = 3x-2$$
$$\frac{du}{dx} = 3x^{1-1} - 0$$
$$= 3x^0$$
$$= 3(1)$$
$$= 3$$

Example 2 (Continued):

Step 3: Substitute the derivatives of y and u into the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \left(\frac{4}{3}u^{1/3}\right) \cdot (3)$$

$$= 4u^{1/3}$$

Step 4: Substitute g(x) back in for u

$$\frac{dy}{dx} = 4u^{1/3}$$

$$= 4(3x - 2)^{1/3}$$

The alternate form of the chain rule shows the derivative in terms of the composite functions.

Chain Rule (Alternate form)

If y = f [g(x)] then the derivative of y is

$$\frac{dy}{dx} = f'\left[g(x)\right] \cdot g'(x)$$

The steps in finding the derivative are similar to those used with the substitution method.

- First separate the composite function into two simpler functions, f and g.
- Second, you would find the derivative of f(x) and g(x).
- Third, substitute g(x) into the derivative of f.
- Last, multiply the result by the derivative of g.

Example 3: Find the derivative of $y = \left(2x^2 - 7x\right)^5$.

Solution:

Step 1: Separate y into two simpler functions.

Hint: Let the function inside the parenthesis be g(x).
Replace g(x) with x to find f(x).

Let $g(x) = 2x^2 - 7x$

Replace g(x) with x

$$y = \left(2x^2 - 7x\right)^5$$

$$y = \left[g(x)\right]^5$$

$$f(x) = x^5$$

Step 2: Find the derivatives of f(x) and g(x)

$$f(x) = x^5 \qquad g(x) = 2x^2 - 7x$$

$$f'(x) = 5x^{5-1} \qquad g'(x) = 2 \cdot 2x^{2-1} - 7x^{1-1}$$

$$= 5x^4 \qquad = 4x - 7$$

Step 3: Substitute g(x) into the derivative of f(x)

$$f'(x) = 5x^4$$

$$f'\left[g(x)\right] = 5\left[g(x)\right]^4$$

$$f'\left[g(x)\right] = 5\left(2x^2 - 7x\right)^4$$

Step 4: Multiply the result from step 3 by the derivative of g(x)

$$f'\left[g(x)\right] \cdot g'(x) = 5\left(2x^2 - 7x\right)^4 \cdot \left(4x - 7\right)$$

$$\frac{dy}{dx} = 5\left(2x^2 - 7x\right)^4 \cdot \left(4x - 7\right)$$

In special cases where a function is raised to a real number exponent (like in the previous two examples) the derivative can be found by using the generalized power rule. The generalized power rule is simply a special case of the chain rule.

Remember back to the Power rule that you have used to find derivatives of functions in the form of x^n. The derivative would be determined by multiplying the coefficient times the exponent and then subtracting one from the exponent.

$$y = cx^n$$
$$y' = c \cdot nx^{n-1}$$

Now lets look at a composite function where a function is raised to an exponent and see what happens.

Assume we are given the function $y = \left[g(x) \right]^n$

If we were to express y as the composition of two functions, we would have g(x) and $f(x) = x^n$.

Applying the chain rule would give us:

$$f(x) = x^n$$
$$f'(x) = nx^{n-1}$$
$$f'[g(x)] = n[g(x)]^{n-1}$$

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x)$$
$$= n[g(x)]^{n-1} \cdot g'(x)$$

Comparing this to the power rule, you will see that the only difference is the multiplication of $g'(x)$

$$n \cdot x^{n-1}$$
$$n \cdot [g(x)]^{n-1} \cdot g'(x)$$

Generalized Power Rule

If g(x) is a function of x and y = $[g(x)]^n$ for any real number n, then the derivative is found by subtracting 1 from the exponent on g(x) and then multiplying it by n and the derivative of g(x).

$$\frac{d}{dx}\left[g(x)\right]^n = n\cdot\left[g(x)\right]^{n-1}\cdot g'(x)$$

Example 4: Find the derivative of $y=\left(x^4+5x^2\right)^{3/2}$ using the generalized power rule.

Solution:

$$y=\left(x^4+5x^2\right)^{3/2}$$

$$\frac{dy}{dx}=\frac{3}{2}\cdot\left(x^4+5x^2\right)^{3/2-1}\cdot D_x\left(x^4+5x^2\right)$$

$$=\frac{3}{2}\cdot\left(x^4+5x^2\right)^{3/2-2/2}\cdot\left(4x^{4-1}+5\cdot 2x^{2-1}\right)$$

$$=\frac{3}{2}\cdot\left(x^4+5x^2\right)^{1/2}\cdot\left(4x^3+10x\right)$$

$$=\frac{3}{2}\cdot\left(x^4+5x^2\right)^{1/2}\cdot 2\cdot\left(2x^3+5x\right)$$

$$=3\left(2x^3+5x\right)\left(x^4+5x^2\right)^{1/2}$$

As you saw in the last section, there are some functions that will require you to use a combination of rules in order to find the derivative. The next example will demonstrate such a case.

Example 5: Find the derivative of $r\left(t\right)=\frac{5t^2-7t}{\left(3t+1\right)^3}$.

Solution:

In order to find the derivative of this function you must apply the quotient rule and the generalized power rule. Since the exponent of 3 is applied only to the denominator, you would begin by applying the quotient rule first.

Example 5 (Continued):

$$r(t)=\frac{5t^2-7t}{(3t+1)^3}$$

$$r'(t)=\frac{(3t+1)^3\cdot D_x\left(5t^2-7t\right)-\left(5t^2-7t\right)\cdot D_x(3t+1)^3}{\left[(3t+1)^3\right]^2}$$

Now you can find the derivative of the functions in the numerator.

$$\begin{aligned}r'(t)&=\frac{(3t+1)^3\cdot D_x\left(5t^2-7t\right)-\left(5t^2-7t\right)\cdot D_x(3t+1)^3}{\left[(3t+1)^3\right]^2}\\&=\frac{(3t+1)^3\cdot\left(5\cdot 2t^{2-1}-7\cdot t^{1-1}\right)-\left(5t^2-7t\right)\cdot 3\cdot(3t+1)^{3-1}\cdot D_x(3t+1)}{(3t+1)^6}\\&=\frac{(3t+1)^3\cdot(10t-7)-\left(5t^2-7t\right)\cdot 3\cdot(3t+1)^2\cdot\left(3\cdot t^{1-1}+0\right)}{(3t+1)^6}\\&=\frac{(3t+1)^3\cdot(10t-7)-\left(5t^2-7t\right)\cdot 3\cdot(3t+1)^2\cdot(3)}{(3t+1)^6}\end{aligned}$$

The last step is to now simplify the derivative.

$$\begin{aligned}r'(t)&=\frac{(3t+1)^3\cdot(10t-7)-\left(5t^2-7t\right)\cdot 3\cdot(3t+1)^2\cdot(3)}{(3t+1)^6}\\&=\frac{(3t+1)^2\left[(3t+1)(10t-7)-\left(5t^2-7t\right)(9)\right]}{(3t+1)^6}\\&=\frac{\left[\left(30t^2-21t+10t-7\right)-\left(45t^2-63t\right)\right]}{(3t+1)^4}\end{aligned}$$

Example 5 (Continued):

$$r'(t) = \frac{30t^2 - 11t - 7 - 45t^2 + 63t}{(3t+1)^4}$$

$$= \frac{-15t^2 + 52t - 7}{(3t+1)^4}$$

$$= -\frac{15t^2 - 52t + 7}{(3t+1)^4}$$