Differentials

This section will show how differentials can be used for linear approximation, marginal analysis, and error estimation. In previous math courses you learned that the slope of a line is found by dividing the change in y by the change in x.

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

When derivatives were introduced you saw that the derivative was equal to the slope of the tangent line at a given value for x. Therefore, since both of these represent the slope of a line there values would be approximately the same when the change in x (Δx) is very close to zero.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$$

In previous sections when you were taking the derivative of a function it was shown that the derivative notations $f'(x)$ and $\frac{dy}{dx}$ were interchangeable. However, now if we set these equal to each other and multiply by *dx* we can obtain the equation for the differential of y.

$$\frac{dy}{dx} = f'(x)$$

$$dx \cdot \frac{dy}{dx} = f'(x) \cdot dx$$

$$dy = f'(x) \cdot dx$$

Example 1: Find the differential of y (*dy*) for the given values of *x* and *dx*.

$y = 8 - x^2 + x^3$; $x = -1$, $dx = 0.02$

Solution:

Step 1: Find the derivative (dy/dx) of the function.

$$y = 8 - x^2 + x^3$$
$$\frac{dy}{dx} = -2x + 3x^2$$

Step 2: Multiply by *dx* to isolate *dy*

$$\frac{dy}{dx} = -2x + 3x^2$$
$$dx \cdot \frac{dy}{dx} = \left(-2x + 3x^2\right) \cdot dx$$
$$dy = \left(-2x + 3x^2\right) \cdot dx$$

Step 3: Substitute in the given values and simplify

$$dy = \left(-2x + 3x^2\right) \cdot dx$$
$$= \left(-2(-1) + 3(-1)^2\right) \cdot (0.02)$$
$$= (2 + 3) \cdot (0.02)$$
$$= 0.1$$

One application of differentials is in linear approximation. You can use differentials to approximate the value of a square root, for example, if a calculator is not available. Looking at the slope formula again you can see that the change in y is equal to the difference in the y values. Lets say we have the two points (x, f(x)) and ($x + \Delta x$, $f(x + \Delta x)$). Then the change in y would be equal to:

$$\Delta y = f(x + \Delta x) - f(x)$$

At the beginning of this section we say how $\Delta y \approx dy$ when Δx is very close to zero. So we can replace Δy with dy in the equation above.

$$dy = f(x + \Delta x) - f(x)$$

We would now add f(x) to both sides to isolate $f(x + \Delta x)$.

$$f(x) + dy \approx f(x + \Delta x)$$

Finally to arrive at the linear approximation formula we would replace the differential dy with its formula of $f'(x)dx$.

$$f(x) + f'(x)dx = f(x + \Delta x)$$

Example 2: Approximate $\sqrt{37}$ using linear approximation.

Solution:

Step 1: Define our function and values for x and Δx

Since we are dealing with a square root the function would be

$$f(x) = \sqrt{x}$$

Next we need to split 37 as a perfect square plus a number. The closest perfect square to 37 would be 36. Therefore

$$x = 36$$
$$dx = \Delta x = 37 - 36 = 1$$

Step 2: Find the derivative of the function f(x)

$$f(x) = \sqrt{x}$$
$$f(x) = x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2x^{\frac{1}{2}}}$$

Step 3: Multiply by dx to isolate dy

$$dx \cdot \frac{dy}{dx} = \frac{1}{2x^{\frac{1}{2}}} \cdot dx$$
$$dy = \frac{1}{2x^{\frac{1}{2}}} \cdot dx$$

Example 2 (Continued):

Step 4: Evaluate dy at the values for x and dx

$$\begin{aligned} dy &= \frac{1}{2x^{\frac{1}{2}}} \cdot dx \\ &= \frac{1}{2(36)^{\frac{1}{2}}} \cdot (1) \\ &= \frac{1}{2(6)} \cdot (1) \\ &= \frac{1}{12} \end{aligned}$$

Step 5: Determine the approximate value of $\sqrt{37}$

$$\begin{aligned} \sqrt{37} &= f\left(x+\Delta x\right) \\ &\approx f\left(x\right) + dy \\ &\approx \sqrt{x} + dy \\ &\approx \sqrt{36} + \frac{1}{12} \\ &\approx 6\frac{1}{12} \end{aligned}$$

When you first learned the concept of derivatives you saw how the derivative could be used to determine the marginal cost (revenue or profit). The marginal cost, for example, tells you approximately how much it will cost to produce the next unit. We can now also use the concept of differentials to perform marginal analysis.

At the beginning of this section, you learned that if y = f(x) then $dy = f'(x) \cdot dx$. If we use C = C(x) as our function then the differential $dC = C'(x) \cdot dx = C'(x) \cdot \Delta x$. Next since ΔC ≈ dC and ΔC = C(x + Δx) – C(x)

$$\begin{aligned} dC &= C'(x) \cdot \Delta x \\ \Delta C &\approx C'(x) \cdot \Delta x \\ C(x+\Delta x) - C(x) &\approx C'(x) \cdot \Delta x \end{aligned}$$

Now if we are trying to determine the cost of producing the next unit then the change in production (Δx) would be 1.

$$C(x+\Delta x) - C(x) \approx C'(x) \cdot \Delta x$$
$$C(x+1) - C(x) \approx C'(x) \cdot (1)$$
$$C(x+1) - C(x) \approx C'(x)$$

It should be noted that the use of C(x + 1) – C(x) as an approximation of the marginal cost is good only if Δx is relatively small when compared to x.

Example 3: If the cost function for a company is defined by $C(x) = x^3 - 2x + 150$ find the marginal cost using both ΔC and $C'(x)$ when:

a) x = 5 and Δx = 1
b) x = 75 and Δx = 1

Solution:

a) Step 1: Find the derivative of the cost function

$$C(x) = x^3 - 2x + 150$$
$$C'(x) = 3x^2 - 2$$

Step 2: Find ΔC when x = 5 and Δx = 1

$$\begin{aligned} C(x+\Delta x) - C(x) &= C(5+1) - C(5) \\ &= C(6) - C(5) \\ &= \left((6)^3 - 2(6) + 150\right) - \left((5)^3 - 2(5) + 150\right) \\ &= (216 - 12 + 150) - (125 - 10 + 150) \\ &= (216 + 138) - (125 + 140) \\ &= 354 - 265 \\ &= 89 \end{aligned}$$

Example 3 (Continued):

Step 3: Find $C'(x)$ when x = 5

$$C'(x) = 3x^2 - 2$$
$$C'(5) = 3(5)^2 - 2$$
$$= 3(25) - 2$$
$$= 75 - 2$$
$$= 73$$

b) Step 1: Find ΔC when x = 75 and Δx = 1

$$C(x+\Delta x) - C(x) = C(75+1) - C(75)$$
$$= C(76) - C(75)$$
$$= \left((76)^3 - 2(76) + 150\right) - \left((75)^3 - 2(75) + 150\right)$$
$$= (438976 - 152 + 150) - (421875 - 150 + 150)$$
$$= (438976 - 2) - (421875)$$
$$= 438974 - 421875$$
$$= 17099$$

Step 2: Find $C'(x)$ when x = 75

$$C'(x) = 3x^2 - 2$$
$$C'(75) = 3(75)^2 - 2$$
$$= 3(5625) - 2$$
$$= 16875 - 2$$
$$= 16873$$

Now after looking at this problem you might be thinking that there seems to be a bigger difference between the numbers when x was 75.

a) ΔC = 89 and $C'(x)$ = 73; 89 – 73 = 16

b) ΔC = 17099 and $C'(x)$ = 16873; 17099 – 16873 = 226

But if we convert these differences into percentages of the marginal cost $C'(x)$ you can see that the percentage gets smaller and we receive a much better approximation when x is a large number compared to Δx.

a) $\dfrac{\Delta C - C'(x)}{C'(x)} = \dfrac{16}{73} \approx 0.219 \;\; or \;\; 21.9\%$

b) $\dfrac{\Delta C - C'(x)}{C'(x)} = \dfrac{226}{16873} \approx 0.013 \;\; or \;\; 1.3\%$

The last part of these notes will deal with how differentials are used in error estimation. In these problems you will still be utilizing the differential formula that we found at the beginning of this section, $dy = f'(x) \cdot dx$.

Let's say that you were looking to determine what the maximum error would be for the area of a square given the maximum error for the length of the sides. The area of the square is given by the formula $A = x^2$ where x is the length of the sides. Therefore, dx would be the maximum error for the length of the sides and dA would be the approximation of the maximum error for the area of the square.

Example 4: The edge of a square is measured as 2.350 centimeters, with a possible error of ± 0.0625 centimeters. Estimate the maximum error in the area of the square.

Solution:

Step 1: Find the derivative of the area function

$$A = x^2$$
$$dA = 2x \cdot dx$$

Step 2: Now substitute in the values for x and dx

$$\begin{aligned} dA &= 2x \cdot dx \\ dA &= 2(2.350)(\pm 0.0625) \\ &= \pm 0.29375 \\ &\approx \pm 0.294 \end{aligned}$$

The maximum error in the area would be approximately 0.294 cm^2.