Higher-Order Derivatives

Higher-order derivatives are simply the derivative of a derivative. You would use the same derivative rules that you learned for finding the first derivative of a function. The higher derivatives will give you the rate of change of a derivative. For example, the second derivative will give you the rate of change of the first derivative.

The notation used for higher derivatives is similar to that used for the first derivative. The table below shows some of the most commonly used derivative notations.

First derivative	$f'(x)$	$\frac{dy}{dx}$	$D_x\left[f(x)\right]$
Second derivative	$f''(x)$	$\frac{d^2y}{dx^2}$	$D_x^2\left[f(x)\right]$
Third derivative	$f'''(x)$	$\frac{d^3y}{dx^3}$	$D_x^3\left[f(x)\right]$
Fourth derivative	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$D_x^4\left[f(x)\right]$

Example 1: Find the second derivative of $f(x) = 2x^3 + 3x^2 - 12x + 5$.

Solution:

Step 1: Find the first derivative.

The function is a polynomial function so you can find the derivative by using the Sum/Difference rule, the Power rule and the Constant rule.

$$\begin{aligned} f(x) &= 2x^3 + 3x^2 - 12x + 5 \\ f'(x) &= D_x\left(2x^3\right) + D_x\left(3x^2\right) - D_x(12x) + D_x(5) \\ &= 6x^{3-1} + 6x^{2-1} - 12x^{1-1} + 0 \\ &= 6x^2 + 6x - 12 \end{aligned}$$

Example 1 (Continued):

Step 2: Find the second derivative.

The first derivative is also a polynomial function, so you would apply the same derivative rules to find the second derivative.

$$\begin{aligned} f'(x) &= 6x^2 + 6x - 12 \\ f''(x) &= D_x\left(6x^2\right) + D_x\left(6x\right) - D_x\left(12\right) \\ &= 12x^{2-1} + 6x^{1-1} - 0 \\ &= 12x + 6 \end{aligned}$$

Example 2: Find the second derivative of $g(x) = \dfrac{\ln(3x)}{2x^2}$

Solution:

Step 1: Find the first derivative.

Finding the derivative of the function g(x) will require the use of multiple derivative rules. Since the function is in the form of a fraction you must begin by applying the quotient rule. However, finding the derivatives of the numerator and denominator will require the use of the Logarithm rule and the Power rule.

Apply the quotient rule

$$g(x) = \frac{\ln(3x)}{2x^2}$$

$$g'(x) = \frac{\left(2x^2\right)D_x\left[\ln(3x)\right] - \ln(3x)D_x\left(2x^2\right)}{\left[2x^2\right]^2}$$

Example 2 (Continued):

Apply the Logarithm rule

$$g'(x)=\frac{\left(2x^2\right)D_x\left[\ln\left(3x\right)\right]-\ln\left(3x\right)D_x\left(2x^2\right)}{\left[2x^2\right]^2}$$

$$=\frac{\left(2x^2\right)\left(\dfrac{D_x\left(3x\right)}{3x}\right)-\ln\left(3x\right)D_x\left(2x^2\right)}{\left[2x^2\right]^2}$$

$$=\frac{\left(2x^2\right)\left(\dfrac{3}{3x}\right)-\ln\left(3x\right)D_x\left(2x^2\right)}{\left[2x^2\right]^2}$$

Apply the Power rule

$$g'(x)=\frac{\left(2x^2\right)\left(\dfrac{3}{3x}\right)-\ln\left(3x\right)D_x\left(2x^2\right)}{\left[2x^2\right]^2}$$

$$=\frac{\left(2x^2\right)\left(\dfrac{3}{3x}\right)-\left[\ln\left(3x\right)\right]\left(4x^{2-1}\right)}{\left[2x^2\right]^2}$$

$$=\frac{\left(2x^2\right)\left(\dfrac{3}{3x}\right)-\left[\ln\left(3x\right)\right]\left(4x\right)}{\left[2x^2\right]^2}$$

Simplify

$$g'(x)=\frac{\left(2x^2\right)\left(\dfrac{3}{3x}\right)-\left[\ln\left(3x\right)\right]\left(4x\right)}{\left[2x^2\right]^2}$$

$$=\frac{\left(2x^2\right)\left(\dfrac{1}{x}\right)-\left[\ln\left(3x\right)\right]\left(4x\right)}{4x^4}$$

$$=\frac{\left(2x\right)-\left(4x\right)\ln\left(3x\right)}{4x^4}$$

Example 2 (Continued):

$$g'(x) = \frac{(2x) - (4x)\ln(3x)}{4x^4}$$
$$= \frac{(2x)\left[1 - 2\ln(3x)\right]}{4x^4}$$
$$= \frac{1 - 2\ln(3x)}{2x^3}$$

Step 2: Find the second derivative

Apply the Quotient rule

$$g'(x) = \frac{1 - 2\ln(3x)}{2x^3}$$
$$g''(x) = \frac{(2x^3)D_x\left[1 - 2\ln(3x)\right] - \left[1 - 2\ln(3x)\right]D_x(2x^3)}{\left[2x^3\right]^2}$$

Apply the Sum/Difference rule

$$g''(x) = \frac{(2x^3)D_x\left[1 - 2\ln(3x)\right] - \left[1 - 2\ln(3x)\right]D_x(2x^3)}{\left[2x^3\right]^2}$$
$$= \frac{(2x^3)\left[D_x(1) - 2D_x\left[\ln(3x)\right]\right] - \left[1 - 2\ln(3x)\right]D_x(2x^3)}{\left[2x^3\right]^2}$$

Apply the Logarithm rule

$$g''(x) = \frac{(2x^3)\left[D_x(1) - 2D_x\left[\ln(3x)\right]\right] - \left[1 - 2\ln(3x)\right]D_x(2x^3)}{\left[2x^3\right]^2}$$
$$= \frac{(2x^3)\left[0 - 2\left(\frac{D_x(3x)}{3x}\right)\right] - \left[1 - 2\ln(3x)\right]D_x(2x^3)}{\left[2x^3\right]^2}$$
$$= \frac{(2x^3)\left[-2\left(\frac{3}{3x}\right)\right] - \left[1 - 2\ln(3x)\right]D_x(2x^3)}{\left[2x^3\right]^2}$$

Example 2 (Continued):

Apply the Power rule

$$g''(x) = \frac{\left(2x^3\right)\left[-2\left(\frac{3}{3x}\right)\right] - \left[1-2\ln(3x)\right] D_x\left(2x^3\right)}{\left[2x^3\right]^2}$$

$$= \frac{\left(2x^3\right)\left[-2\left(\frac{3}{3x}\right)\right] - \left[1-2\ln(3x)\right]\left(6x^{3-1}\right)}{\left[2x^3\right]^2}$$

$$= \frac{\left(2x^3\right)\left[-2\left(\frac{3}{3x}\right)\right] - \left[1-2\ln(3x)\right]\left(6x^2\right)}{\left[2x^3\right]^2}$$

Simplify

$$g''(x) = \frac{\left(2x^3\right)\left[-2\left(\frac{3}{3x}\right)\right] - \left[1-2\ln(3x)\right]\left(6x^2\right)}{\left[2x^3\right]^2}$$

$$= \frac{\left(2x^3\right)\left(-\frac{2}{x}\right) - \left[6x^2 - 12x^2\ln(3x)\right]}{4x^6}$$

$$= \frac{-4x^2 - 6x^2 + 12x^2\ln(3x)}{4x^6}$$

$$= \frac{-10x^2 + 12x^2\ln(3x)}{4x^6}$$

$$= \frac{-2x^2\left[5 - 6\ln(3x)\right]}{4x^6}$$

$$= -\frac{5 - 6\ln(3x)}{2x^4}$$

An example of the application of higher derivatives in a real world environment can be seen in the area of physics. The distance that an object travels over a period of time can be given by a function named s(t). The first derivative, $s'(t)$, would represent the rate of change of the distance or its velocity. The velocity is normally denoted in function notation as v(t). Now if we were to take the derivative of the velocity function, $v'(t)$, you would have the rate of change of the velocity or acceleration, denoted as a(t).

Therefore the relationship between distance, velocity, and acceleration can be summarized as:

Distance = $s(t)$

Velocity = $v(t) = s'(t)$

Acceleration = $a(t) = v'(t) = s''(t)$

Example 3: The distance s (in feet) that a car travels after t seconds is given by the function $s(t) = t^3 – 2t^2 – 7t + 9$ where $0 \leq t \leq 6$. Find the velocity at any time t, the acceleration at any time t, and the acceleration 4 seconds after the car starts moving.

Solution:

Step 1: Find v(t) - the first derivative of s(t)

$s(t) = t^3 – 2t^2 – 7t + 9$
$s'(t) = 3t^2 – 4t – 7$

$v(t) = 3t^2 – 4t – 7$

Step 2: Find a(t) - the second derivative of s(t)

$s'(t) = 3t^2 – 4t – 7$
$s''(t) = 6t – 4$

$a(t) = 6t – 4$

Step 3: Evaluate a(t) at t = 4

$a(t) = 6t – 4$
$a(4) = 6(4) – 4$
$a(4) = 24 – 4$
$a(4) = 20$

After 4 seconds the car's acceleration is 20 ft/sec^2.