Marginal Functions in Economics

One of the applications of derivatives in a real world situation is in the area of marginal analysis. Marginal analysis uses the derivative (or rate of change) to determine the rate at which a particular quantity is increasing or decreasing. In this section, the marginal functions that we will cover are those for the cost, average cost, revenue, and profit functions. The last topic that will be covered is the elasticity of demand.

No matter which function we are dealing with, the word "marginal" indicates to us that we need to find the derivative of the function. For example, if we are asked to find the marginal cost function then we need to find the derivative of the cost function. When the marginal function is evaluated it will give the approximate change for the next unit. For example, if we evaluated a marginal cost function when x = 100 then the value of $C'(100)$ would be the approximate cost of producing the next unit (or the 101^{st} unit).

Example 1: If a company's total cost function is defined as $C(x) = 0.00002x^3 - 0.02x^2 + 400x + 50000$, find the marginal cost function and evaluate it when x = 200.

Solution:

Find the first derivative of the cost function

$$C(x) = 0.00002x^3 - 0.02x^2 + 400x + 5000$$
$$C'(x) = 0.00006x^2 - 0.04x + 400$$

Substitute the given value for x into the marginal cost function

$$C'(x) = 0.00006x^2 - 0.04x + 400$$
$$C'(200) = 0.00006(200)^2 - 0.04(200) + 400$$
$$C'(200) = 0.00006(40000) - 0.04(200) + 400$$
$$C'(200) = 2.4 - 8 + 400$$
$$C'(200) = 394.4$$

The approximate cost of producing the 201^{st} unit will be $394.40.

The next function that you may be asked to find is the average cost function. The average cost function is determined in the same manner that you would find an average. We want to divide the total production cost, C(x), by the total number of units, x. The average cost function is distinguished from the cost function with a bar above the function notation, $\overline{C}(x)$.

$$\text{Average cost } \overline{C}(x) = \frac{C(x)}{x}$$

The marginal average cost function would then be obtained by taking the first derivative of the average cost function.

Example 2: Using the same cost function from Exercise 1, $C(x) = 0.00002x^3 – 0.02x^2 + 400x + 50000$, find the marginal average cost when x = 200.

Solution:

First, we need to find the average cost function

$$\overline{C}(x) = \frac{C(x)}{x}$$

$$\overline{C}(x) = \frac{0.00002x^3 - 0.02x^2 + 400x + 50000}{x}$$

$$\overline{C}(x) = 0.00002x^2 - 0.02x + 400 + \frac{50000}{x}$$

Next, find the marginal average cost function by taking the 1st derivative of $\overline{C}(x)$

$$\overline{C}(x) = 0.00002x^2 - 0.02x + 400 + \frac{50000}{x}$$

$$\overline{C}(x) = 0.00002x^2 - 0.02x + 400 + 50000x^{-1}$$

$$\overline{C}'(x) = 0.00004x - 0.02 + 0 - 50000x^{-2}$$

$$\overline{C}'(x) = 0.00004x - 0.02 - \frac{50000}{x^2}$$

Now, evaluate the marginal average cost function when x = 200

$$\overline{C}'(x) = 0.00004x - 0.02 - \frac{50000}{x^2}$$

$$\overline{C}'(200) = 0.00004(200) - 0.02 - \frac{50000}{(200)^2}$$

$$\overline{C}'(200) = 0.008 - 0.02 - \frac{50000}{40000}$$

$$\overline{C}'(200) = 0.008 - 0.02 - 1.25$$

$$\overline{C}'(200) = -1.262$$

The process of finding the marginal revenue and marginal profit function is the same as how we found the marginal cost function. The only difference that you may encounter is the need to first determine the revenue or profit functions. If the revenue function is not given, then it will be equal to the price per unit (p) times the number of items sold (x).

$$R(x) = px$$

The profit function is the difference between the revenue and cost functions.

$$P(x) = R(x) - C(x)$$

Example 3: Suppose the relationship between the unit price p in dollars and the quantity demanded x is given by the equation $p = -0.03x + 750$ where $0 \le x \le 25{,}000$. Find and interpret $R'(3000)$.

Solution

First, find the revenue function

$$R(x) = px$$
$$R(x) = (-0.03x + 750)(x)$$
$$R(x) = -0.03x^2 + 750x$$

Next, find the marginal revenue function

$$R(x) = -0.03x^2 + 750x$$
$$R'(x) = -0.06x + 750$$

Now, evaluate the marginal revenue function when $x = 3000$

$$R'(x) = -0.06x + 750$$
$$R'(3000) = -0.06(3000) + 750$$
$$R'(3000) = -180 + 750$$
$$R'(3000) = 570$$

The sale of the 3001st unit would produce revenue of approximately \$570.

Example 4: Suppose a company's weekly demand for their product is $p = 500 - 0.05x$ where p is the unit price in dollars and x is the quantity demanded which must be between 0 and 12000. The cost function is given by $C(x) = 0.00002x^3 - 0.03x^2 + 300x + 78000$. Find $P'(2500)$ and interpret the results.

Solution

First, we need to find the revenue function

$$R(x) = px$$
$$R(x) = (500 - 0.05x)(x)$$
$$R(x) = 500x - 0.05x^2$$

Example 4 (Continued):

Next, find the profit function

$$P(x) = R(x) - C(x)$$
$$P(x) = (500x - 0.05x^2) - (0.00002x^3 - 0.03x^2 + 300x + 78000)$$
$$P(x) = 500x - 0.05x^2 - 0.00002x^3 + 0.03x^2 - 300x - 78000$$
$$P(x) = -0.00002x^3 - 0.02x^2 + 200x - 78000$$

Now, find the marginal profit function

$$P(x) = -0.00002x^3 - 0.02x^2 + 200x - 78000$$
$$P'(x) = -0.00006x^2 - 0.04x + 200$$

Last, evaluate the marginal profit function when x = 2500

$$P'(x) = -0.00006x^2 - 0.04x + 200$$
$$P'(2500) = -0.00006(2500)^2 - 0.04(2500) + 200$$
$$P'(2500) = -0.00006(6250000) - 0.04(2500) + 200$$
$$P'(2500) = -375 - 100 + 200$$
$$P'(2500) = -275$$

The sale of the 2501st unit would result in a loss of approximately $275.

The last topic for this section of notes is the elasticity of demand. The elasticity of demand represents a ratio of the percentage change in quantity demanded to the percentage change in the unit price.

$$E(p) = \frac{pf'(p)}{f(p)}$$

There are three possible cases for the elasticity of demand:

1. Elastic – when E(p) > 1

   A small percentage change in unit price will result in a larger percentage change in quantity demanded. Revenue is decreasing on an interval where the demand is elastic.

2. Unitary – when E(p) = 1

   A small percentage change in unit price will result in the same percentage change in quantity demanded. Revenue is constant at the point where the demand is unitary.

3. Inelastic – when $E(p) < 1$

   A small percentage change in unit price will result in a smaller percentage change in quantity demanded. Revenue is increasing on an interval where the demand is inelastic.

Example 5: Compute the elasticity of demand for the demand equation, $4x + 5p = 80$, and determine whether the demand is elastic, unitary, or inelastic when the price is \$10.

Solution

Solve the demand equation in terms of x

$$4x + 5p = 80$$
$$4x = -5p + 80$$
$$x = -\frac{5}{4}p + 20$$

let $f(p) = x$

$$f(p) = -\frac{5}{4}p + 20$$

Find the derivative of the demand function f(p)

$$f(p) = -\frac{5}{4}p + 20$$
$$f'(p) = -\frac{5}{4}$$

Find the elasticity of demand

$$E(p) = \frac{pf'(p)}{f(p)}$$
$$E(p) = \frac{p(-\frac{5}{4})}{-\frac{5}{4}p + 20}$$
$$E(p) = \frac{p(-\frac{5}{4})}{-\frac{5}{4}p + 20} * \frac{4}{4}$$
$$E(p) = \frac{-5p}{-5p + 80}$$

Exercise 5 (Continued):

$$E(p) = \frac{p}{p-16}$$

Evaluate the elasticity of demand when p = 10

$$E(p) = \frac{p}{p-16}$$

$$E(10) = \frac{10}{10-16}$$

$$E(10) = \frac{10}{-6}$$

$$E(10) = -\frac{5}{3} \approx -1.7$$

Since the elasticity of demand is < 1, the demand is inelastic when p = 10.