The Product and Quotient Rules

In this section, you will learn how to find the derivative of a product of functions and the derivative of a quotient of functions.

A function that is the product of functions will be in the form of $f(x) = u(x) \cdot v(x)$. In the last section you saw that the derivative of the sum of functions was equal to the sum of the derivatives. The derivative of a product of functions however does not equal to the product of the derivatives. To find the derivative of a product of functions you would use the following derivative rule.

Product Rule

If $f(x) = u(x) \cdot v(x)$ and the derivatives of u and v both exist, then the derivative of f is equal to:

$$f'(x) = \frac{d}{dx}\left[u(x) \cdot v(x)\right] = u(x) \cdot v'(x) + v(x) \cdot u'(x)$$

As you can see, the derivative is equal to the first function times the derivative of the second plus the second function times the derivative of the first.

When you need to find the derivative of a product of functions begin by finding the derivatives of the individual functions. Then substitute them into the Product Rule. Lets look at a few examples of how to use the Product Rule.

Example 1: Find the derivative of $f(x) = (4x^2)(3x-2)$.

Solution:

Step 1: Identify the individual functions and find their derivatives

$$u(x) = 4x^2 \qquad v(x) = 3x - 2$$

$$u'(x) = 4 \cdot 2x^{2-1} \qquad v'(x) = 3 \cdot x^{1-1} - 0$$

$$u'(x) = 8x \qquad v'(x) = 3$$

Example 1 (Continued):

Step 2: Substitute the functions and their derivatives into the Product Rule

$$f'(x) = u(x)\cdot v'(x) + v(x)\cdot u'(x)$$
$$= \left(4x^2\right)\cdot(3) + (3x-2)\cdot(8x)$$
$$= 12x^2 + 24x^2 - 16x$$
$$= 36x^2 - 16x$$

Note: This is the same answer you would get if you were to multiply the two functions “$4x^2$” and “3x – 2” together and then find the derivative using the rules covered in the last section.

$$f(x) = \left(4x^2\right)(3x-2)$$
$$= 12x^3 - 8x^2$$
$$f'(x) = 12\cdot 3x^{3-1} - 8\cdot 2x^{2-1}$$
$$= 36x^2 - 16x$$

In the first example, the derivative could be found without using the Product Rule if the functions are multiplied first. However, there will be times when multiplying the functions will not be easy and the Product Rule is necessary. The next example will show a case where this is true.

Example 2: Find the derivative of $g(x) = \left(2\sqrt[3]{x^2} - 3x\right)\left(6x^3 - x^2 + 5\right)$.

Solution:

Step 1: Rewrite the radical in exponential form $\sqrt[m]{x^n} = x^{n/m}$

$$g(x) = \left(2\sqrt[3]{x^2} - 3x\right)\left(6x^3 - x^2 + 5\right)$$
$$= \left(2x^{2/3} - 3x\right)\left(6x^3 - x^2 + 5\right)$$

Example 2 (Continued):

Step 2: Identify the individual functions and find their derivatives

$$u(x) = 2x^{2/3} - 3x$$

$$u'(x) = 2 \cdot \frac{2}{3} x^{2/3 - 1} - 3 \cdot x^{1-1}$$

$$= \frac{4}{3} x^{2/3 - 3/3} - 3 \cdot 1$$

$$= \frac{4}{3} x^{-1/3} - 3$$

$$v(x) = 6x^3 - x^2 + 5$$

$$v'(x) = 6 \cdot 3x^{3-1} - 1 \cdot 2x^{2-1} + 0$$

$$= 18x^2 - 2x$$

Step 3: Substitute the functions and their derivatives into the Product Rule

$$g'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$$

$$= \left(2x^{2/3} - 3x\right)\left(18x^2 - 2x\right) + \left(6x^3 - x^2 + 5\right)\left(\frac{4}{3} x^{-1/3} - 3\right)$$

Step 4: Simplify by multiplying and combining like terms

$$g'(x) = \left(2x^{2/3} - 3x\right)\left(18x^2 - 2x\right) + \left(6x^3 - x^2 + 5\right)\left(\frac{4}{3} x^{-1/3} - 3\right)$$

$$= 2x^{2/3}\left(18x^2 - 2x\right) - 3x\left(18x^2 - 2x\right) + 6x^3\left(\frac{4}{3} x^{-1/3} - 3\right) - x^2\left(\frac{4}{3} x^{-1/3} - 3\right) + 5\left(\frac{4}{3} x^{-1/3} - 3\right)$$

$$= 36x^{2/3+2} - 4x^{2/3+1} - 54x^3 + 6x^2 + \frac{24}{3} x^{3-1/3} - 18x^3 - \frac{4}{3} x^{2-1/3} + 3x^2 + \frac{20}{3} x^{-1/3} - 15$$

$$= 36x^{8/3} - 4x^{5/3} - 54x^3 + 6x^2 + 8x^{8/3} - 18x^3 - \frac{4}{3} x^{5/3} + 3x^2 + \frac{20}{3} x^{-1/3} - 15$$

$$= (-54 - 18)x^3 + (36 + 8)x^{8/3} + (6 + 3)x^2 + (-4 - \frac{4}{3})x^{5/3} + \frac{20}{3} x^{-1/3} - 15$$

$$= -72x^3 + 44x^{8/3} + 9x^2 - \frac{16}{3} x^{5/3} + \frac{20}{3} x^{-1/3} - 15$$

$$= -72x^3 + 44x^{8/3} + 9x^2 - \frac{16}{3} x^{5/3} + \frac{20}{3x^{1/3}} - 15$$

You now know how to find the derivative of functions that involve the sum, difference, or product of functions. The next case that we will look at is where you are asked to find the derivative of a rational function, which is the quotient of two functions. These functions will be in the form of

$$f(x) = \frac{u(x)}{v(x)}$$

In order to find the derivative of a rational function, the derivatives of the numerator & denominator must exist and the function in the denominator must not be equal to zero, v(x) ≠ 0. As long as these conditions are met then the derivative of a rational function can be determined by the following formula called the Quotient Rule.

Quotient Rule

If $f(x) = \frac{u(x)}{v(x)}$, where all derivatives exists and v(x) is not zero, then the derivative of f(x) is:

$$f'(x) = \frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{\left[v(x)\right]^2}$$

The derivative of a rational function is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator all divided by the denominator squared.

Example 3: Find the derivative of $q(x) = \frac{2x^3 - 5x^2}{x+2}$

Solution:

Step 1: Find the derivative of the numerator and denominator.

$$u(x) = 2x^3 - 5x^2 \qquad v(x) = x + 2$$

$$u'(x) = 2\cdot 3x^{3-1} - 5\cdot 2x^{2-1} \qquad v'(x) = 1\cdot x^{1-1} + 0$$

$$= 6x^2 - 10x \qquad = 1$$

Example 3 (Continued):

Step 2: Substitute the functions and their derivatives into the Quotient Rule

$$
\begin{aligned}
q'(x) &= \frac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{\left[v(x)\right]^2} \\
&= \frac{(x+2)\cdot\left(6x^2-10x\right) - \left(2x^3-5x^2\right)\cdot(1)}{(x+2)^2}
\end{aligned}
$$

Step 3: Simplify by multiplying and combining like terms

$$
\begin{aligned}
q'(x) &= \frac{(x+2)\cdot\left(6x^2-10x\right) - \left(2x^3-5x^2\right)\cdot(1)}{(x+2)^2} \\
&= \frac{(x)\cdot\left(6x^2-10x\right) + (2)\cdot\left(6x^2-10x\right) - \left(2x^3-5x^2\right)}{(x+2)^2} \\
&= \frac{6x^3-10x^2+12x^2-20x-2x^3+5x^2}{(x+2)^2} \\
&= \frac{(6-2)x^3+(-10+12+5)x^2-20x}{(x+2)^2} \\
&= \frac{4x^3+7x^2-20x}{(x+2)^2}
\end{aligned}
$$

If you do not want to use the quotient rule, then you could rewrite the function so that it is the product of two functions. This could be done by bringing the denominator up into the numerator with a negative exponent.

Lets look at an example where the derivative of a rational expression can be found by first rewriting it so that it is in the form of a product of two functions.

Example 4: Find the derivative of $k(x) = \dfrac{2x^3 - 5x^2}{x^4}$ using the Product Rule.

Solution:

Step 1: Rewrite the rational expression as a product of two functions

$$\begin{aligned} k(x) &= \frac{2x^3 - 5x^2}{x^4} \\ &= \left(2x^3 - 5x^2\right)\left(x^{-4}\right) \end{aligned}$$

Step 2: Identify the individual functions and find their derivatives

$$\begin{aligned} u(x) &= 2x^3 - 5x^2 \\ u'(x) &= 2 \cdot 3x^{3-1} - 5 \cdot 2x^{2-1} \\ &= 6x^2 - 10x \end{aligned}$$

$$\begin{aligned} v(x) &= x^{-4} \\ v'(x) &= -4x^{-4-1} \\ &= -4x^{-5} \end{aligned}$$

Step 3: Substitute the functions and their derivatives into the Product Rule

$$\begin{aligned} k'(x) &= u(x) \cdot v'(x) + v(x) \cdot u'(x) \\ &= \left(2x^3 - 5x^2\right)\left(-4x^{-5}\right) + \left(x^{-4}\right)\left(6x^2 - 10x\right) \end{aligned}$$

Step 4: Simplify by multiplying and combining like terms

$$\begin{aligned} k'(x) &= \left(2x^3 - 5x^2\right)\left(-4x^{-5}\right) + \left(x^{-4}\right)\left(6x^2 - 10x\right) \\ &= 2x^3\left(-4x^{-5}\right) - 5x^2\left(-4x^{-5}\right) + \left(x^{-4}\right)\left(6x^2\right) + \left(x^{-4}\right)\left(-10x\right) \\ &= -8x^{3-5} + 20x^{2-5} + 6x^{-4+2} - 10x^{-4+1} \\ &= -8x^{-2} + 20x^{-3} + 6x^{-2} - 10x^{-3} \\ &= (-8 + 6)x^{-2} + (20 - 10)x^{-3} \\ &= -2x^{-2} + 10x^{-3} \\ &= -\frac{2}{x^2} + \frac{10}{x^3} \end{aligned}$$

Just as with the product rule there will be times where you will be better of using the quotient rule instead of rewriting the problem. Also keep in mind that in some examples you may be required to use a combination of derivative rules in order to get your answer. But if you take the derivative one step at a time you will be able to find the derivative without too much trouble.

The next example will illustrate a situation where more than one derivative rule must be applied in order to get the derivative of the given function.

Example 5: Find the derivative of $h(t)=\dfrac{(8t^2-1)(5t-2)}{(2t+3)}$.

Solution:

Step 1: In this problem you have a product within a quotient. Therefore you would first apply the quotient rule.

$$h'(t)=\frac{v(t)\cdot u'(t)-u(t)\cdot v'(t)}{\left[v(t)\right]^2}$$

Since the numerator is a product you must apply the product rule to find the derivative of u(t). We will use f and g to identify the two functions.

$$u(t)=(8t^2-1)(5t-2)$$

$$f(t)=8t^2-1 \qquad g(t)=5t-2$$

$$f'(t)=8\cdot 2t^{2-1}-0 \qquad g'(t)=5\cdot 1t^{1-1}-0$$

$$=18t \qquad =5$$

$$\begin{aligned} u'(t)&=f(t)\cdot g'(t)+g(t)\cdot f'(t)\\ &=(8t^2-1)(5)+(5t-2)(18t)\\ &=40t^2-5+90t^2-36t\\ &=130t^2-36t-5 \end{aligned}$$

The derivative of the denominator would be

$$\begin{aligned} v(t)&=2t+3\\ v'(t)&=2\cdot 1t^{1-1}+0\\ &=2 \end{aligned}$$

Example 5 (Continued):

Step 2: Substitute the functions and their derivatives into the Quotient Rule

$$h'(t) = \frac{v(t)\cdot u'(t) - u(t)\cdot v'(t)}{\left[v(t)\right]^2}$$

$$= \frac{(2t+3)\cdot(130t^2 - 36t - 5) - (8t^2 - 1)(5t-2)\cdot(2)}{(2t+3)^2}$$

Step 3: Simplify by multiplying and combining like terms

$$h'(t) = \frac{(2t+3)\cdot(130t^2 - 36t - 5) - (8t^2 - 1)(5t-2)\cdot(2)}{(2t+3)^2}$$

$$= \frac{(2t+3)\cdot(130t^2 - 36t - 5) - (8t^2 - 1)(10t-4)}{(2t+3)^2}$$

$$= \frac{(260t^3 - 72t^2 - 10t + 390t^2 - 108t - 15) - (80t^3 - 32t^2 - 10t + 4)}{(2t+3)^2}$$

$$= \frac{(260t^3 + 318t^2 - 118t - 15) - (80t^3 - 32t^2 - 10t + 4)}{(2t+3)^2}$$

$$= \frac{(260-80)t^3 + (318+32)t^2 + (-118+10)t + (-15-4)}{(2t+3)^2}$$

$$= \frac{180t^3 + 350t^2 - 108t - 19}{(2t+3)^2}$$