

Differentiation of Exponential Functions

The next derivative rules that you will learn involve exponential functions. An exponential function is a function in the form of a constant raised to a variable power. The variable power can be something as simple as “ x ” or a more complex function such as “ $x^2 - 3x + 5$ ”.

Basic Exponential Function

$$y = b^x, \text{ where } b > 0 \text{ and not equal to } 1$$

Exponential Function with a function as an exponent

$$y = b^{g(x)}$$

The derivative of an exponential function would be determined by the use of the chain rule, which was covered in the previous section. In reviewing the derivative rules for exponential functions we will begin by looking at the derivative of a function with the constant raised to a simple variable.

Derivative of an exponential function in the form of $y = b^x$

If $y = b^x$ where $b > 0$ and not equal to 1 then the derivative is equal to the original exponential function multiplied by the natural log of the base.

$$y' = (\ln b)b^x$$

Example 1: Find the derivative of $y = 5^x$.

Solution:

Since you have a constant raised to the variable x , the derivative would be equal to the original function multiplied by the natural log of the base, which is 5.

$$y = 5^x$$

$$y' = (\ln 5)5^x$$

This derivative rule can be simplified when the base of the exponential function is equal to “ e ”. The derivative involves the natural log of the base. However, if the base is equal to “ e ” then the

natural log of the base can be reduced to the value of 1. (See our [logarithms formula sheet](#) for a full list of logarithm properties.) This would simplify the derivative to the original function itself.

$$\begin{aligned}y &= e^x \\y' &= (\ln e)e^x \\&= (1)e^x \\&= e^x\end{aligned}$$

Derivative of an exponential function in the form of $y = e^x$

If $y = e^x$ then the derivative is simply equal to the original function of e^x .

Example 2: Find the derivative of $y = e^u$.

Solution:

Since the base of the exponential function is equal to “e” the derivative would be equal to the original function.

$$\begin{aligned}y &= e^u \\y' &= e^u\end{aligned}$$

Now lets say you are given the function $y = b^{g(x)}$ and are asked to find its derivative. In this case, you will need to use the chain rule to determine the derivative. To see how the chain rule would be used we will rewrite this function as the composition of the functions $f(u)$ and $g(x)$.

We will begin by letting u equal the exponent of $g(x)$

$$u = g(x)$$

$$\begin{aligned}y &= b^{g(x)} \\y &= b^u\end{aligned}$$

Now we will let $f(u)$ equal y

$$f(u) = y$$

$$y = b^u$$

$$f(u) = b^u$$

Therefore, $y = f[g(x)] = b^{g(x)}$.

Now recalling the chain rule from the last section we can determine the rule for finding the derivative of an exponential function.

Chain rule:

$$\text{If } y = f[g(x)] \text{ then } \frac{dy}{dx} = f'[g(x)] \cdot g'(x)$$

In the chain rule formula, $f'[g(x)] = f'(u)$ for this problem. So what we need to do is find the derivative of $f(u)$ and $g(x)$.

$$f(u) = b^u$$

$$f'(u) = (\ln b)b^u$$

Now substituting $g(x)$ back for u gives us:

$$f'(u) = (\ln b)b^u$$

$$f'[g(x)] = (\ln b)b^{g(x)}$$

Substituting this derivative into the chain rule formula will then give you the derivative rule for finding the derivative of exponential functions.

$$y = f[g(x)] = b^{g(x)}$$

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x)$$

$$= (\ln b)b^{g(x)} \cdot g'(x)$$

As with the previous derivative rules for exponential functions, this rule can be simplified to $\frac{dy}{dx} = e^{g(x)} \cdot g'(x)$ if the base is equal to “e”.

Derivatives of $b^{g(x)}$ and $e^{g(x)}$

$$\text{If } y = b^{g(x)} \text{ then } y' = \frac{dy}{dx} = (\ln b)b^{g(x)} \cdot g'(x)$$

$$\text{If } y = e^{g(x)} \text{ then } y' = \frac{dy}{dx} = e^{g(x)} \cdot g'(x)$$

Example 3: Find the derivative of $y = 5^{-2x^3}$

Solution:

Here you have a constant raised to a function so you will use the derivative rule

$$y' = (\ln b)b^{g(x)} \cdot g'(x)$$

$$y = 5^{-2x^3}$$

$$y' = (\ln b)b^{g(x)} \cdot g'(x)$$

$$= (\ln 5)5^{-2x^3} \cdot D_x(-2x^3)$$

$$= (\ln 5)5^{-2x^3} \cdot (-2 \cdot 3x^{3-1})$$

$$= (\ln 5)5^{-2x^3} \cdot (-6x^2)$$

$$= -6x^2(\ln 5)5^{-2x^3}$$

Example 4: Find the derivative of $y = -6e^{3x}$

Solution:

Since the base of the exponential function in this problem is “e” you can use the derivative rule $y' = e^{g(x)} \cdot g'(x)$

$$\begin{aligned}y &= -6e^{3x} \\y' &= e^{g(x)} \cdot g'(x) \\&= -6e^{3x} \cdot D_x(3x) \\&= -6e^{3x} \cdot (3) \\&= -18e^{3x}\end{aligned}$$

Example 5: Find the derivative of $y = -4x^3 e^{2x^2}$

Solution:

This problem involves the product of two functions, one a power function and the other an exponential function. To find the derivative you will have to apply a combination of the product rule, the power rule, and the exponential rule.

Step 1: Apply the product rule.

$$\begin{aligned}y &= -4x^3 e^{2x^2} \\y' &= (-4x^3) \cdot D_x(e^{2x^2}) + (e^{2x^2}) \cdot D_x(-4x^3)\end{aligned}$$

Step 2: Apply the power and exponential rules.

$$\begin{aligned}y' &= (-4x^3) \cdot D_x(e^{2x^2}) + (e^{2x^2}) \cdot D_x(-4x^3) \\&= (-4x^3) \cdot [e^{2x^2} \cdot D_x(2x^2)] + (e^{2x^2}) \cdot (-4 \cdot 3x^{3-1}) \\&= (-4x^3) \cdot [e^{2x^2} \cdot (2 \cdot 2x^{2-1})] + (e^{2x^2}) \cdot (-12x^2) \\&= (-4x^3) \cdot (e^{2x^2}) \cdot (4x) + (e^{2x^2}) \cdot (-12x^2)\end{aligned}$$

Example 5 (Continued):

Step 3: Simplify the derivative

$$\begin{aligned}y' &= (-4x^3) \cdot (e^{2x^2}) \cdot (4x) + (e^{2x^2}) \cdot (-12x^2) \\&= (-16x^4) \cdot (e^{2x^2}) + (e^{2x^2}) \cdot (-12x^2) \\&= -4x^2 e^{2x^2} (4x^2 + 3)\end{aligned}$$