Differentiation of Logarithmic Functions

The rule for finding the derivative of a logarithmic function is given as:

If $y = \log_a x$ then $\dfrac{dy}{dx}$ *or* $y' = \dfrac{1}{(\ln a)x}$.

This rule can be proven by rewriting the logarithmic function in exponential form and then using the exponential derivative rule covered in the last section.

$y = \log_a x$	Begin with logarithmic function
$a^y = x$	Convert into exponential form
$(\ln a)\cdot a^y \cdot y' = 1$	Differentiate both sides of the equation
$(\ln a)\cdot x \cdot y' = 1$	Substitute x for the exponential function a^y
$y' = \dfrac{1}{(\ln a)x}$	Solve for y´ by dividing each side by "(ln a)x"

As with the exponential rules, the derivative of a logarithmic function can be simplified if the base of the logarithm is "e".

$y = \log_e x = \ln x$	Logarithmic function with base "e"
$y' = \dfrac{1}{(\ln e)x}$	Apply the logarithm derivative rule above
$y' = \dfrac{1}{(1)x}$	Use properties of logarithms to simplify
$y' = \dfrac{1}{x}$	

Example 1: Find the derivative of $f(x) = \log_6 x$

Solution:

$$f(x) = \log_6 x$$
$$f'(x) = \frac{1}{(\ln 6)x}$$

Example 2: Find the derivative of $f(x) = \log(4x^2)$

Solution:

Right now the only derivative rule we have for logarithms is for "log x". So in order to take the derivative of this function we must first use the properties of logarithms to rewrite the function.

First use the product property of $\log(xy) = \log x + \log y$

$$\begin{aligned} f(x) &= \log(4x^2) \\ &= \log(4) + \log(x^2) \end{aligned}$$

Next use the exponent property of $\log(x^r) = r \cdot \log(x)$

$$\begin{aligned} f(x) &= \log(4) + \log(x^2) \\ &= \log(4) + 2 \cdot \log(x) \end{aligned}$$

Now we can find the derivative.

$$\begin{aligned} f(x) &= \log(4) + 2 \cdot \log(x) \\ f'(x) &= 0 + 2 \cdot \frac{1}{(\ln 10)x} \\ &= \frac{2}{(\ln 10)x} \end{aligned}$$

Even though we were able to find the derivative of this function, it would be easier to find the derivative if we had a rule that dealt with the situation where a function is equal to the log of another function. In order to do this, we would have to combine the chain rule with the logarithm rule.

Lets look at an example similar to the first function that we looked at when proving the logarithm derivative rule.

$f\left[g(x)\right] = \log_b\left[g\left(x\right)\right]$	Composite logarithmic function
$b^{f\left[g(x)\right]} = g\left(x\right)$	Convert to exponential form
$\left(\ln b\right)\cdot b^{f\left[g(x)\right]}\cdot f'\left[g(x)\right] = g'\left(x\right)$	Differentiate both sides of the equation
$\left(\ln b\right)\cdot g\left(x\right)\cdot f'\left[g(x)\right] = g'\left(x\right)$	Substitute g(x) for the exponential function
$f'\left[g(x)\right] = \dfrac{g'\left(x\right)}{\left(\ln b\right)\cdot g\left(x\right)}$	Divide each side by "(ln b) g(x)"

Derivative of $\log_b\left[g\left(x\right)\right]$

If $y = \log_b\left[g\left(x\right)\right]$ then $y' = \dfrac{1}{\ln b}\cdot\dfrac{g'\left(x\right)}{g\left(x\right)}$

If $y = \ln\left[g\left(x\right)\right]$ then $y' = \dfrac{g'\left(x\right)}{g\left(x\right)}$

Now lets go back to example 2 and find the derivative again but this time using the above rule.

Example 3: Find the derivative of $f(x) = \log(4x^2)$

Solution:

Note: Remember if a base is not shown it is understood to be 10.

$$
\begin{aligned}
f(x) &= \log(4x^2) \\
f'(x) &= \frac{1}{\ln 10} \cdot \frac{D_x(4x^2)}{4x^2} \\
&= \frac{1}{\ln 10} \cdot \frac{8x}{4x^2} \\
&= \frac{1}{\ln 10} \cdot \frac{2}{x} \\
&= \frac{2}{(\ln 10)x}
\end{aligned}
$$

As you can see we get the same derivative as before but in an shorter and easier process.

Example 4: Find the derivative of $y = \ln(t^4 + 5t)^{3/4}$

Solution:

In this problem we can simplify the derivative process by first applying the properties of logarithms to move the exponent out in front of the function as a coefficient.

$$
\begin{aligned}
y &= \ln(t^4 + 5t)^{3/4} \\
&= \frac{3}{4}\ln(t^4 + 5t)
\end{aligned}
$$

Now we can find the derivative

$$y = \frac{3}{4}\ln(t^4 + 5t)$$

$$y' = \frac{3}{4} \cdot D_x\left[\ln(t^4 + 5t)\right]$$

Example 4 (Continued):

$$y' = \frac{3}{4} \cdot \frac{D_x\left(t^4 + 5t\right)}{\left(t^4 + 5t\right)}$$

$$y' = \frac{3}{4} \cdot \frac{\left(4t^3 + 5\right)}{\left(t^4 + 5t\right)}$$

Now lets look at a more complex function that will require the use of several derivative rules. Remember when finding the derivative of a complex function take it step by step. Don't try to do it all at once.

Example 5: Find the derivative of $y = \dfrac{\left(x^2 + 1\right)e^{2x}}{\ln\left(5x + 3\right)}$

Solution:

Since the function is in the form of a fraction we must begin by applying the quotient rule. When you go to find the derivative of the numerator you are will have to use both the product and exponential rules. The derivative of the denominator will require the use of the logarithm rule.

First, apply the quotient rule.

$$y = \frac{\left(x^2 + 1\right)e^{2x}}{\ln\left(5x + 3\right)}$$

$$y' = \frac{\ln\left(5x + 3\right) \cdot D_x\left[\left(x^2 + 1\right)e^{2x}\right] - \left(x^2 + 1\right)e^{2x} \cdot D_x \ln\left(5x + 3\right)}{\left[\ln\left(5x + 3\right)\right]^2}$$

Example 5 (Continued):

Next, apply the product and exponential rule to the derivative of the numerator.

$$y' = \frac{\ln(5x+3)\cdot D_x\left[\left(x^2+1\right)e^{2x}\right] - \left(x^2+1\right)e^{2x}\cdot D_x \ln(5x+3)}{\left[\ln(5x+3)\right]^2}$$

$$= \frac{\ln(5x+3)\cdot\left[\left(x^2+1\right)\cdot D_x e^{2x} + e^{2x}\cdot D_x\left(x^2+1\right)\right] - \left(x^2+1\right)e^{2x}\cdot D_x \ln(5x+3)}{\left[\ln(5x+3)\right]^2}$$

$$= \frac{\ln(5x+3)\cdot\left[\left(x^2+1\right)\cdot\left[e^{2x}\cdot D_x(2x)\right] + e^{2x}\cdot(2x)\right] - \left(x^2+1\right)e^{2x}\cdot D_x \ln(5x+3)}{\left[\ln(5x+3)\right]^2}$$

$$= \frac{\ln(5x+3)\cdot\left[\left(x^2+1\right)\left(2e^{2x}\right) + 2xe^{2x}\right] - \left(x^2+1\right)e^{2x}\cdot D_x \ln(5x+3)}{\left[\ln(5x+3)\right]^2}$$

Now, apply the logarithmic rule to find the derivative of the denominator.

$$y' = \frac{\ln(5x+3)\cdot\left[\left(x^2+1\right)\left(2e^{2x}\right) + 2xe^{2x}\right] - \left(x^2+1\right)e^{2x}\cdot D_x \ln(5x+3)}{\left[\ln(5x+3)\right]^2}$$

$$= \frac{\ln(5x+3)\cdot\left[\left(x^2+1\right)\left(2e^{2x}\right) + 2xe^{2x}\right] - \left(x^2+1\right)e^{2x}\cdot \frac{D_x(5x+3)}{(5x+3)}}{\left[\ln(5x+3)\right]^2}$$

$$= \frac{\ln(5x+3)\cdot\left[\left(x^2+1\right)\left(2e^{2x}\right) + 2xe^{2x}\right] - \left(x^2+1\right)e^{2x}\cdot \frac{5}{(5x+3)}}{\left[\ln(5x+3)\right]^2}$$

The only thing left to do now is to simplify the derivative using the properties of algebra.

Example 5 (Continued):

$$y' = \frac{\ln(5x+3)\cdot\left[\left(x^2+1\right)\left(2e^{2x}\right)+2xe^{2x}\right]-\left(x^2+1\right)e^{2x}\cdot\frac{5}{(5x+3)}}{\left[\ln(5x+3)\right]^2}$$

$$= \frac{2e^{2x}\left(x^2+x+1\right)\ln(5x+3)-\left(x^2+1\right)e^{2x}\cdot\frac{5}{(5x+3)}}{\left[\ln(5x+3)\right]^2}$$

$$= \frac{2e^{2x}\left(x^2+x+1\right)(5x+3)\ln(5x+3)-\left(x^2+1\right)5e^{2x}}{(5x+3)\left[\ln(5x+3)\right]^2}$$

$$= \frac{e^{2x}\left[2\left(x^2+x+1\right)(5x+3)\ln(5x+3)-5\left(x^2+1\right)\right]}{(5x+3)\left[\ln(5x+3)\right]^2}$$