Review Exercise Set 18

Exercise 1: Find the derivative of the given logarithmic function.

\[ f(x) = x \ln (x^3 - 3x)^2 \]

Exercise 2: Find the derivative of the given logarithmic function.

\[ y = \frac{2x^2 - 5x + 3}{\ln (x^3 - 1)} \]

Exercise 3: Find the derivative of the given logarithmic function.

\[ s(t) = \ln [(2t + 1)^3(5t^2 - t)] \]
Exercise 4: Find the derivative of the given logarithmic function.

\[ y = \left(e^{2u-3} + 1\right)\left(\ln(3u + 4)^3\right) \]

Exercise 5: Find the number of units that need to be manufactured so that the profit is maximized given the following revenue and cost functions.

\[ R(x) = \ln (3x + 1)^{25} ; C(x) = 5x + 13 \]
Exercise 1: Find the derivative of the given logarithmic function.

\[ f(x) = x \ln (x^3 - 3x)^2 \]

Apply the logarithmic properties to simplify the expression

\[ f(x) = 2x \ln (x^3 - 3x) \]

Find derivative of each term

\[ D_x(2x) = 2 \]

\[ D_x \ln (x^3 - 3x) = \frac{D_x(x^3 - 3x)}{x^3 - 3x} = \frac{3x^2 - 3}{x^3 - 3x} \]

Substitute derivatives into the product rule

\[ f'(x) = 2x \cdot D_x \ln (x^3 - 3x) + \ln (x^3 - 3x) \cdot D_x(2x) \]

\[ f'(x) = 2x \cdot \frac{3x^2 - 3}{x^3 - 3x} + \ln (x^3 - 3x) \]

\[ f'(x) = 2x \cdot \frac{3x^2 - 3}{x(x^2 - 3)} + 2 \ln (x^3 - 3x) \]

\[ f'(x) = \frac{6x^2 - 6}{x^2 - 3} + \ln (x^3 - 3x)^2 \]

Exercise 2: Find the derivative of the given logarithmic function.

\[ y = \frac{2x^2 - 5x + 3}{\ln (x^3 - 1)} \]

Find derivative of each term

\[ D_x(2x^2 - 5x + 3) = 4x - 5 \]

\[ D_x(\ln (x^3 - 1)) = \frac{D_x(x^3 - 1)}{x^3 - 1} = \frac{3x^2}{x^3 - 1} \]

\[ D_x(\ln (x^3 - 1)) = \frac{3x^2}{x^3 - 1} \]
Exercise 2 (Continued):

Substitute derivatives into the quotient rule

\[
y' = \frac{\ln (x^3 - 1) D_x \left( 2x^2 - 5x + 3 \right) - \left( 2x^2 - 5x + 3 \right) D_x \ln (x^3 - 1)}{\left[ \ln (x^3 - 1) \right]^2}
\]

\[
= \frac{\ln (x^3 - 1) \left( 4x - 5 \right) - \left( 2x^2 - 5x + 3 \right) \frac{3x^2}{x^3 - 1}}{\left[ \ln (x^3 - 1) \right]^2}
\]

\[
= \frac{\ln (x^3 - 1) \left( 4x - 5 \right) - \left( 2x^2 - 5x + 3 \right) \frac{3x^2}{x^3 - 1} \times \frac{x^3 - 1}{x^3 - 1}}{\left[ \ln (x^3 - 1) \right]^2}
\]

\[
= \frac{(4x - 5)(x^3 - 1) \ln (x^3 - 1) - \left( 2x^2 - 5x + 3 \right) \left( 3x^2 \right)}{(x^3 - 1) \left[ \ln (x^3 - 1) \right]^2}
\]

Exercise 3: Find the derivative of the given logarithmic function.

\[ s(t) = \ln [(2t + 1)^3(5t^2 - t)] \]

Apply the logarithmic properties to simplify the equation

\[ s(t) = \ln (2t + 1)^3 + \ln (5t^2 - t) \]
\[ s(t) = 3 \ln (2t + 1) + \ln (5t - 1) \]
\[ s(t) = 3 \ln (2t + 1) + \ln (5t - 1) \]

Take the derivative of each logarithmic term

\[
s'(t) = 3 \times \frac{D_t (2t + 1)}{2t + 1} + \frac{D_t (t)}{t} + \frac{D_t (5t - 1)}{5t - 1}
\]

\[
= 3 \times \frac{2}{2t + 1} + \frac{1}{t} + \frac{5}{5t - 1}
\]

\[
= \frac{6}{2t + 1} + \frac{1}{t} + \frac{5}{5t - 1}
\]
Exercise 4: Find the derivative of the given logarithmic function.

\[ y = \left( e^{2u-3} + 1 \right) \left( \ln \sqrt{3u+4} \right)^3 \]

Find the derivative of each term

\[ D_u (e^{2u-3} + 1) = e^{2u-3} * D_u (2u - 3) + 0 \]
\[ D_u (e^{2u-3} + 1) = e^{2u-3} * (2) \]
\[ D_u (e^{2u-3} + 1) = 2e^{2u-3} \]

\[ D_u \left( \ln \sqrt{3u+4} \right)^3 = 3 \left( \ln \sqrt{3u+4} \right)^2 \frac{D_u \left( \ln \sqrt{3u+4} \right)}{\left( \frac{3u+4}{2} \right)^{1/2}} \]
\[ = 3 \left( \ln \sqrt{3u+4} \right)^2 \frac{1}{2} \left( \frac{3u+4}{2} \right)^{1/2} (3) \]
\[ = 3 \left( \ln \sqrt{3u+4} \right)^2 \frac{3}{2} \left( \frac{3u+4}{2} \right) \]
\[ = \left( \ln \sqrt{3u+4} \right)^2 \frac{9}{2(3u+4)} \]

Apply the product rule to the function

\[ y' = \left( e^{2u-3} + 1 \right) D_u \left( \ln \sqrt{3u+4} \right)^3 + \left( \ln \sqrt{3u+4} \right)^3 D_u \left( e^{2u-3} + 1 \right) \]

Substitute in the derivatives of the terms

\[ y' = \left( e^{2u-3} + 1 \right) \left( \ln \sqrt{3u+4} \right)^2 \frac{9}{2(3u+4)} + \left( \ln \sqrt{3u+4} \right)^3 \left( 2e^{2u-3} \right) \]
Exercise 5: Find the number of units that need to be manufactured so that the profit is maximized given the following revenue and cost functions.

\[ R(x) = \ln (3x + 1)^{25} \; ; \; C(x) = 5x + 13 \]

Find the profit function

\[ P(x) = R(x) - C(x) \]
\[ P(x) = \ln (3x + 1)^{25} - (5x + 13) \]
\[ P(x) = 25 \ln (3x + 1) - 5x - 13 \]

Find the marginal profit function

\[ P'(x) = 25 \cdot \frac{D_x (3x + 1)}{3x + 1} - 5 - 0 \]
\[ P'(x) = 25 \cdot \frac{3}{3x + 1} - 5 \]
\[ P'(x) = \frac{75}{3x + 1} - 5 \]

Set marginal profit function equal to zero and solve for \( x \)

\[ 0 = \frac{75}{3x + 1} - 5 \]
\[ 5 = \frac{75}{3x + 1} \]
\[ 5(3x + 1) = 75 \]
\[ 15x + 5 = 75 \]
\[ 15x = 70 \]
\[ x = 4.67 \]

Find second derivative

\[ P''(x) = 75(3x + 1)^{-1} - 5 \]
\[ P''(x) = -75(3x + 1)^{-2}(3) - 0 \]
\[ P''(x) = -\frac{225}{(3x + 1)^2} \]
Exercise 5 (Continued):

Evaluate second derivative with critical number to test for absolute maximum

\[ P''(4.67) = -\frac{225}{(3(4.67)+1)^2} \]

\[ P''(4.67) \approx -1 \]

Since the second derivative is concave down 4.67 is the location that will maximize the profit function

Determine number of units to produce

The critical number is a decimal but the company cannot produce a fractional unit so we would evaluate the profit function at 4 and 5 units to determine which will maximize the profit.

\[ P(4) = 25 \ln (3(4) + 1) - 5(4) - 13 \]
\[ P(4) = 25 \ln (13) - 20 - 13 \]
\[ P(4) = 25 \ln (13) - 33 \]
\[ P(4) \approx 31.12 \]

\[ P(5) = 25 \ln (3(5) + 1) - 5(5) - 13 \]
\[ P(5) = 25 \ln (16) - 25 - 13 \]
\[ P(5) = 25 \ln (16) - 38 \]
\[ P(5) \approx 31.31 \]

The number of units to be produce to maximize the profit function is 5 units.