One-sided Limits and Continuity

A limit written in the form of $\lim_{x \to a} f(x)$ is called a two-sided limit. This means that x is approaching the number "a" from both sides (from the left and from the right). However, there may be times when you only want to find the limit from one side. To do this you would use one-sided limits.

One-sided limits are denoted by placing a positive (+) or negative (-) sign as an exponent on the value "a". For example, if you wanted to find a one-sided limit from the left then the limit would look like $\lim_{x \to a^-} f(x)$. This limit would be read as "the limit of f(x) as x approaches a from the left."

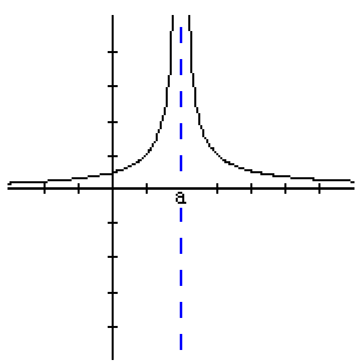
A right-handed limit would look like $\lim_{x \to a^+} f(x)$ and would be read as "the limit of f(x) as x approaches a from the right."

Finding one-sided limits are important since they will be used in determining if the two-sided limit exists. For the two-sided limit to exist both one-sided limits must exist and be equal to the same value.

$$\lim_{x \to a} f(x) \text{ exists if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = M \text{ and L = M.}$$

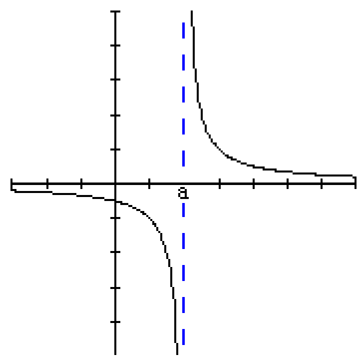
The following three cases are situations where the limit of f as x approaches a may not exist.

1. If f(x) approaches infinity (either positive or negative) as x approaches a from either side, then the limit $\lim_{x \to a} f(x)$ does not exist.



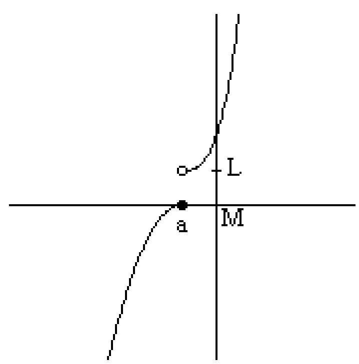
$\lim_{x \to a^-} f(x) = \infty \quad \lim_{x \to a^+} f(x) = \infty$

2. If f(x) approaches positive infinity as x approaches a from one side and negative infinity as x approaches a from the other side, then the limit $\lim_{x \to a} f(x)$ does not exist.



$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = \infty$

3. If f(x) approaches the number L from one side and the number M from the other side, then the limit $\lim_{x \to a} f(x)$ does not exist.



$\lim_{x \to a^-} f(x) = M \qquad \lim_{x \to a^+} f(x) = L$

A function is said to be continuous if there is no break (or gap) in the graph over an open interval. If you are able to sketch the graph of a function without having to stop and lift your pencil from the graph then the function is continuous. However, if is not always convenient or possible to quickly sketch the graph of a function to determine if it is continuous at any given point. In order to determine if a function is continuous at a given point you would use the definition of continuity.

Definition of Continuity at x = c

A function f is continuous at x = c if all three of the following conditions are satisfied. If the function fails any one of the three conditions, then the function is discontinuous at x = c.

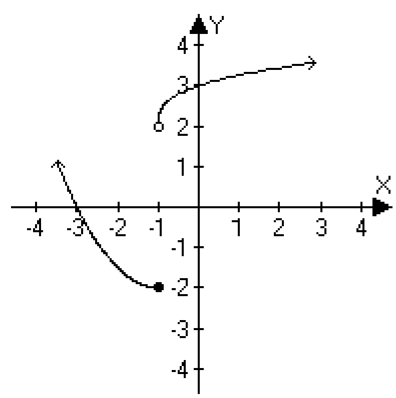
1.) The function must be defined at x = c. f(c) is defined
2.) The limit of the function must exist as x approaches c $\lim_{x \to c} f(x)$ exists
3.) The value of f as x approaches c must be equal to f(c) $\lim_{x \to c} f(x) = f(c)$

Remember that in order for the limit to exist the left-hand and right-hand limits must exist and approach the same value.

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

Lets now look at a few examples to see how this definition is used in determining continuity of a function.

Example 1: Tell why the function in the graph below is discontinuous at x = -1.



Solution:

Start by testing if the first condition is true, f(x) is defined at x = -1.

f(-1) = -2

The function f is defined at x = -1 by the point (-1, -2). Therefore, the function passes the first condition for continuity. Next test to see if the limit exists as x approaches –1.

Example 1 (Continued):

Limit as x approaches –1 from the left is

$$\lim_{x \to -1^-} f(x) = -2$$

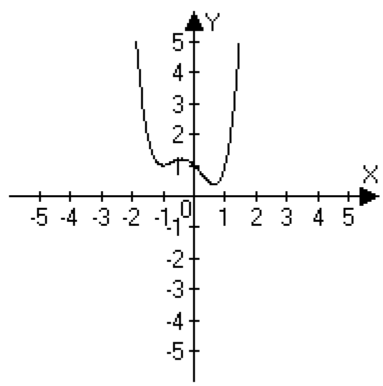
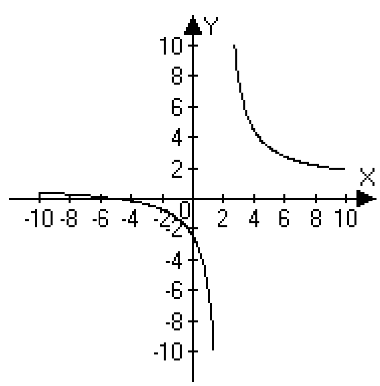
Limit as x approaches –1 from the right is

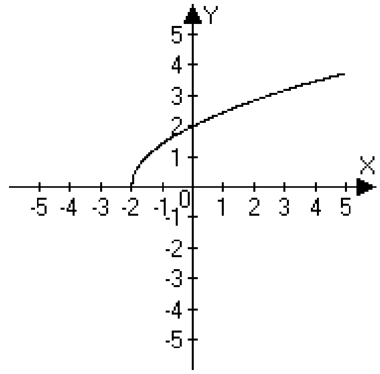
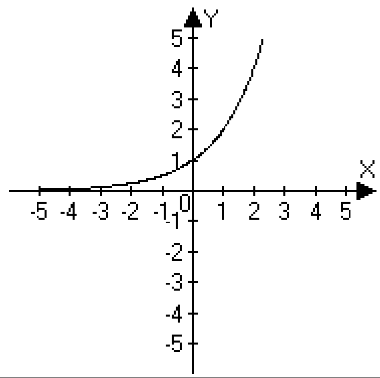
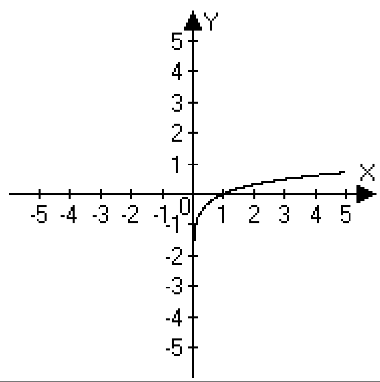
$$\lim_{x \to -1^+} f(x) = 2$$

The left-handed limit does not approach the same number as the right-handed limit ($-2 \neq 2$).

Therefore the function fails the second condition and is discontinuous at x = -1.

It is helpful to remember the characteristics of some of the more common graphs of basic functions. Keeping these characteristics in mind will help speed up the process of determining at what points (if any) a function is discontinuous.

Type of Function	Sample Graph	Continuity
Polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$		Continuous for all values of x
Rational function $f(x) = \frac{p(x)}{q(x)}$		Continuous for all values of x except for those that make the denominator zero

Square Root function $f(x)=\sqrt{ax+b}$		Continuous for all values of x as long as the radicand is greater than or equal to zero.
Exponential function $f(x)=b^x$		Continuous for all values of x
Logarithmic function $f(x)=\log_b x$		Continuous for all values of x greater than zero

Example 2: Find all values of x where the following function is discontinuous.

$$\lim_{x\to 2}\frac{x^2+3x-10}{x-2}$$

Example 2 (Continued):

Solution:

In this example you are taking the limit of a rational function, so the possible point of discontinuity will be what makes the denominator zero.

$$x - 2 = 0$$
$$x = 2$$

Now you will begin applying the three tests for continuity where x = 2.

Test 1: f(2) is defined

$$\begin{aligned} f(x) &= \frac{x^2 + 3x - 10}{x - 2} \\ &= \frac{(x+5)(x-2)}{x-2} \\ &= x + 5 \\ f(2) &= 2 + 5 \\ &= 7 \end{aligned}$$

f(2) = 7 however this point is not included in the graph because if x = 2 then the denominator in the original equation would be equal to zero. Therefore, the function is not defined at x = 2 and fails the first test of continuity.

Since the function fails the first test there is no reason to continue with the other two tests. The function is discontinuous at x = 2.

Example 3: Find all values of x where the following function is discontinuous.

$$f(x) = \begin{cases} 1 - x & if \;\; x < 1 \\ 2 & if \;\; 1 \le x \le 2 \\ 4 - x & if \;\; x > 2 \end{cases}$$

Example 3 (Continued):

Solution:

The possible values of discontinuity for this piecewise function are at 1 and 2. So you would apply the tests of continuity for both of these values. First we will look at x = 1.

Test 1: f(1) is defined

When x is between 1 and 2 we will use the function f(x) = 2.

f(x) = 2
f(1) = 2

f(1) = 2 therefore the function is defined at x = 1 and passes the first test of continuity.

Test 2: $\lim_{x \to 1} f(x)$ *exists*

To determine if the limit exists we will compare the one-sided limits from the left and right to see if they approach the same value. Approaching 1 from the left we will use the function f(x) = 1 – x.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(1 - x)$$
$$= 1 - 1$$
$$= 0$$

Approaching 1 from the right we will use the function f(x) = 2.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(2)$$
$$= 2$$

Compare the one-sided limits

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

$$0 \neq 2$$

The one-sided limits do not approach the same value (0 ≠ 2) therefore the limit of the function as x approaches 1 does not exist. Since the limit does not exist we do not need to perform the third test and we can say the function is discontinuous at x = 1.

Example 3 (Continued):

Solution:

Now we will look at the second possible point of discontinuity, x = 2.

Test 1: f(2) is defined.

When x is between 1 and 2 we will use the function f(x) = 2.

f(x) = 2
f(2) = 2

f(2) = 2 therefore the function is defined at x = 2 and passes the first test of continuity.

Test 2: $\lim_{x \to 2} f(x)$ *exists*

First we will look at the limit approaching 2 from the left. For this limit will use the function f(x) = 2.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2)$$
$$= 2$$

Now for the limit approaching 2 from the right we will use the function f(x) = 4 – x.

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (4 - x)$$
$$= 4 - 2$$
$$= 2$$

Compare the one-sided limits

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$2 = 2$$

The one-sided limits approach the same value (2) therefore the limit of the function as x approaches 2 does exist and is equal to 2. So now we will look at the third and final test for continuity.

Example 3 (Continued):

Solution:

Test 3: $\lim_{x\to 2} f(x) = f(2)$

$$\lim_{x\to 2} f(x) = f(2)$$
$$2 = 2$$

The limit of f(x) as x approaches 2 is equal to the same value as f(2). Therefore the function passes all three tests and is continuous at x = 2.

Intermediate Value Theorem

The intermediate value theorem is one that plays an important part in the discussion of the continuity of a function and locating its zeros. The theorem states:

> If f is a continuous function on a closed interval [a, b] and M is any number between f(a) and f(b), then there is at least one number c in [a, b] such that f(c) = M.

Therefore, if we know that f(a) and f(b) have opposite signs (one positive and one negative) then there must be at least one zero (x-intercept) in the closed interval [a, b].