The Derivative

The Tangent Line

Let two points on the graph of a function, f(x), be (a, f(a)) and (a + h, f(a + h)). The line passing through these points is called the secant line (see figure 1) and its slope is equal to the average rate of change between the two points.

$$\text{Slope of secant line} = \text{average rate of change} = \frac{f(a+h)-f(a)}{h}$$

If we let h approach zero, the point (a + h, f(a + h)) will get closer and closer to the point (a, f(a)), as shown in figure 1. This would then result in giving us the instantaneous rate of change where x = a, which is what is called the slope of the tangent line. Therefore, a tangent line is a line that touches the graph of a function at only one point provided that the limit as h approaches zero of the difference quotient exists.

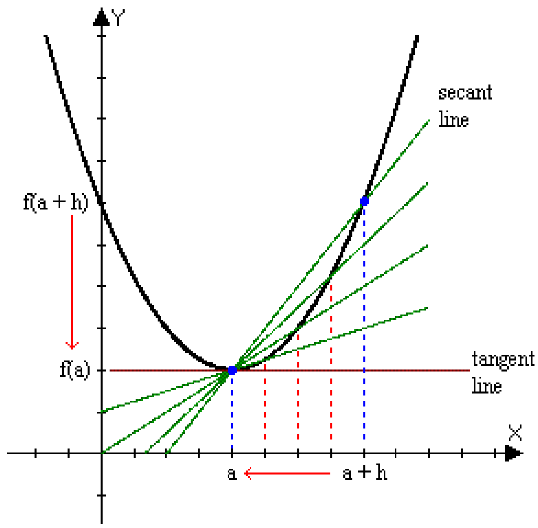

Figure 1

This slope and the point (a, f(a)) can then be substituted into the point-slope form of a line to determine the equation of the tangent line.

Point-slope form of a line

$$y - y_1 = m(x - x_1)$$

where m = slope of tangent line and (x_1, y_1) = the point (a, f(a))

Example 1: Find the slope of the tangent line to the graph of $f(x) = x^2 + 3x - 4$ at $x = 1$. Find the equation of the tangent line.

Solution:

To determine the slope of the tangent you would begin by finding f(1) and f(1 + h).

$f(x) = x^2 + 3x - 4$
$f(1) = (1)^2 + 3(1) - 4$
$f(1) = 1 + 3 - 4$
$f(1) = 0$

$f(x) = x^2 + 3x - 4$
$f(1 + h) = (1 + h)^2 + 3(1 + h) - 4$
$f(1 + h) = 1 + 2h + h^2 + 3 + 3h - 4$
$f(1 + h) = h^2 + 5h$

Now substitute f(1) and f(1 + h) into the formula for the slope of the tangent line.

$$\text{Slope of tangent line} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$\begin{aligned}
\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0} \frac{h^2 + 5h - 0}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 5h}{h} \\
&= \lim_{h \to 0} \frac{h(h+5)}{h} \\
&= \lim_{h \to 0} (h+5) \\
&= 0 + 5 \\
&= 5
\end{aligned}$$

Next, substitute the calculated slope and point into the point-slope form of a line.

$(a, f(a)) = (1, 0)$ and $m = 5$

$y - y_1 = m(x - x_1)$
$y - 0 = 5(x - 1)$
$y = 5x - 5$

The equation of the tangent line to f(x) at the point (1, 0) is $y = 5x - 5$.

The instantaneous rate of change formula can also be used to determine the derivative of a function. The derivative is a function of x which can give us the slope of the tangent line at any given point in the domain of the function, f(x). The derivative is denoted as $f'(x)$ and is read as "f prime of x."

The derivative of the function f(x) at x is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note: In some textbooks you will see them use the symbol Δx (read delta x) instead of the letter h. Both of them mean the same thing but using h is less confusing than Δx, so in these notes you will see the formula with the variable h.

Finding the derivative of a function can be broken down into the following four steps:

1. Find f(x + h)
2. Find and simplify the expression f(x + h) – f(x).

   Note: After you perform this step, the only terms you should be left with are those that contain the variable "h" in them.

3. Divide by h to get the difference quotient, $\frac{f(x+h) - f(x)}{h}$
4. Find the derivative by taking the limit as h approaches 0 of the difference quotient
   $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 2: Find the derivative of the function $f(x) = 2x^2 - 5x - 3$.

Solution:

Step 1: Find f(x + h)

$$\begin{aligned} f(x) &= 2x^2 - 5x - 3 \\ f(x+h) &= 2(x+h)^2 - 5(x+h) - 3 \\ f(x+h) &= 2(x^2 + 2xh + h^2) - 5x - 5h - 3 \\ f(x+h) &= 2x^2 + 4xh + 2h^2 - 5x - 5h - 3 \end{aligned}$$

Step 2: Find and simplify f(x + h) – f(x)

$$\begin{aligned} f(x+h) - f(x) &= (2x^2 + 4xh + 2h^2 - 5x - 5h - 3) - (2x^2 - 5x - 3) \\ &= 2x^2 + 4xh + 2h^2 - 5x - 5h - 3 - 2x^2 + 5x + 3 \\ &= 4xh + 2h^2 - 5h \end{aligned}$$

Example 2 (Continued):

Step 3: Divide by "h"

$$\frac{f(x+h)-f(x)}{h} = \frac{4xh+2h^2-5h}{h}$$

$$= \frac{h(4x+2h-5)}{h}$$

$$= 4x+2h-5$$

Step 4: Find the derivative

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}(4x+2h-5)$$

$$= 4x+2(0)-5$$

$$= 4x-5$$

There are several situations where the derivative of a function will not exist. These included

1. At sharp points or corners
2. At points where the function is discontinuous
3. At vertical asymptotes
4. At points where the slope of the tangent line is undefined (vertical tangent line)
5. At points where the function is undefined