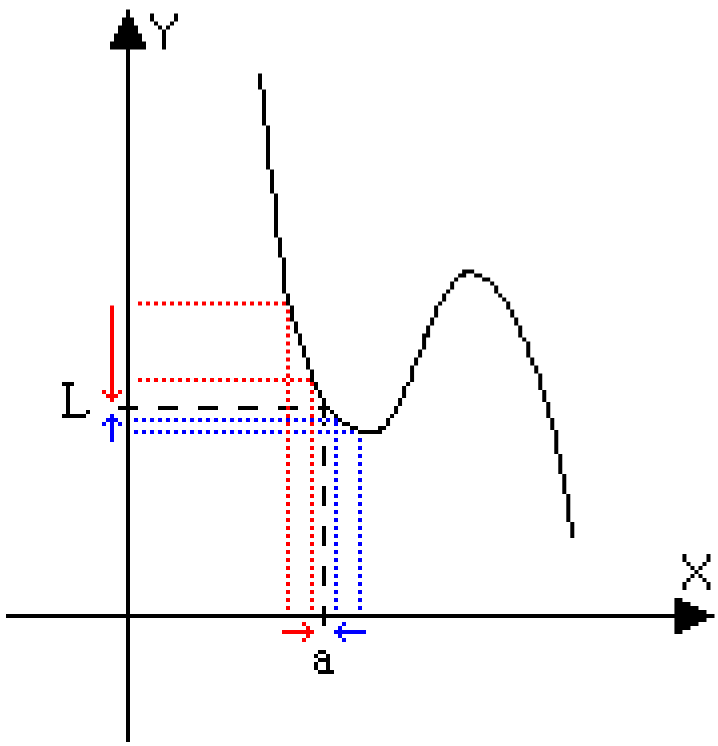
Limits

In this section, we will look at the concept of finding limits. But before we start doing this lets cover some of the definitions and rules for finding limits.

What is a limit? When finding a limit you are attempting to find out if a function (say f(x) for example) approaches a single value as x approaches a given number from either side. This can sometimes be better understood visually.

In the figure below, we have the graph of a function called f(x). As x approaches (or gets closer and closer to) the number “a” from both sides then the value of f(x) approaches the number “L”.



This will then give us the definition for the limit of a function.

Let “f” be a function and “a” and “L” be real numbers. If x gets closer and closer to “a” on both sides and the corresponding values of f(x) get closer and closer to L then L is the limit of f(x) as x approaches a. The limit would be written as:

$$\lim_{x \to a} f(x) = L$$

Finding the limits of simple functions like “3x” or “x – 10” can be done by simply creating a table of values or by graphing the function. However more complex functions would take to long to graph or develop a table of values. Therefore, there are some rules you can use to make finding limits quicker and easier.

Properties of Limits

Let a, L and M be real numbers, and let f and g be functions such that

$$\lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M$$

1. The limit of a constant is equal to the constant.

$$\lim_{x \to a} k = k$$

2. The limit of a constant times a function is equal to the constant times the limit of the function.

$$\begin{aligned}\lim_{x \to a}\left[k \cdot f(x)\right] &= k \cdot \lim_{x \to a} f(x) \\ &= k \cdot L\end{aligned}$$

3. The limit of a sum or difference of functions is equal to the sum or difference of the limits of the functions.

$$\begin{aligned}\lim_{x \to a}\left[f(x) \pm g(x)\right] &= \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \\ &= L + M\end{aligned}$$

4. The limit of the product of functions is equal to the product of the limits of the functions.

$$\begin{aligned}\lim_{x \to a}\left[f(x)g(x)\right] &= \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right] \\ &= LM\end{aligned}$$

5. The limit of a quotient of functions is equal to the quotient of the limits of the functions, as long as the limit of the denominator is not zero.

$$\begin{aligned}\lim_{x \to a} \frac{f(x)}{g(x)} &= \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \\ &= \frac{L}{M}\end{aligned}$$

$M \neq 0$

Properties of Limits (Continued)

6. The limit of a polynomial is equal to the polynomial evaluated at "a".

$$\lim_{x \to a} p(x) = p(a)$$

7. The limit of a function raised to a real number exponent is equal to the limit of the function raised to the exponent (provided the limit exists)

$$\begin{aligned}\lim_{x \to a} \left[f(x) \right]^k &= \left[\lim_{x \to a} f(x) \right]^k \\ &= L^k\end{aligned}$$

Now let's look at some examples using these properties.

Example 1: Suppose that $\lim_{x \to 3} f(x) = 15$ and $\lim_{x \to 3} g(x) = 5$. Use the limit properties to find the following limit.

$$\lim_{x \to 3} \left[f(x) - 2g(x) \right]$$

Solution:

First use property 3 to rewrite the problem as the difference of two limits.

$$\lim_{x \to 3} \left[f(x) - 2g(x) \right] = \lim_{x \to 3} f(x) - \lim_{x \to 3} \left[2g(x) \right]$$

Next use property 2 to move the constant outside of the limit.

$$\lim_{x \to 3} f(x) - \lim_{x \to 3} \left[2g(x) \right] = \lim_{x \to 3} f(x) - 2\lim_{x \to 3} g(x)$$

Substitute in the values of the limits and simplify.

$$\begin{aligned}\lim_{x \to 3} f(x) - 2\lim_{x \to 3} g(x) &= 15 - 2(5) \\ &= 15 - 10 \\ &= 5\end{aligned}$$

Example 2: Suppose that $\lim\limits_{x \to 1} f(x) = 8$ and $\lim\limits_{x \to 1} g(x) = 4$. Use the limit rules to find the following limit.

$$\lim_{x \to 1} \frac{f(x) + g(x)}{3g(x)}$$

Solution:

First use property 5 to take the limit of the numerator and denominator.

$$\lim_{x \to 1} \frac{f(x) + g(x)}{3g(x)} = \frac{\lim\limits_{x \to 1} \left[f(x) + g(x) \right]}{\lim\limits_{x \to 1} \left[3g(x) \right]}$$

Now use property 3 in the numerator and property 2 in the denominator.

$$\frac{\lim\limits_{x \to 1} \left[f(x) + g(x) \right]}{\lim\limits_{x \to 1} \left[3g(x) \right]} = \frac{\lim\limits_{x \to 1} f(x) + \lim\limits_{x \to 1} g(x)}{3 \lim\limits_{x \to 1} g(x)}$$

Substitute in the values for the limits and simplify.

$$\frac{\lim\limits_{x \to 1} f(x) + \lim\limits_{x \to 1} g(x)}{3 \lim\limits_{x \to 1} g(x)} = \frac{8 + 4}{3(4)}$$

$$= \frac{12}{12}$$

$$= 1$$

Indeterminate forms for limits

A limit is said to be in indeterminate form when we have zero over. The property for the quotient of f(x)/g(x) cannot be used to determine the value of the expression in its current form. Therefore, we must examine the expression to find a way of reducing it to an equivalent form which can be evaluated using the properties of limits.

Example 3: Use the rules of limits to determine if the limit exists. If the limit exists, find its value.

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$$

Solution:

Normally you would begin by using property 5 to take the limit of the numerator and denominator but the limit of the denominator would be zero, $\lim_{x \to 4}(x-4) = 4 - 4 = 0$. Therefore, you must begin by reducing the fraction. To do this you would factor the numerator and simplify the resulting fraction.

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} \frac{(x-4)(x+4)}{x-4}$$
$$= \lim_{x \to 4}(x+4)$$

Now you can use property 6 to find the limit.

$$\lim_{x \to 4}(x+4) = 4 + 4$$
$$= 8$$

In addition to finding the limit of a function as x approaches a single number, we can also find the limit of a function as x approaches infinity (either negative or positive). Finding the limit as x approaches infinity will provide us with the horizontal asymptote (if one exists) for the function. In order for a horizontal asymptote to exist, the limit of a function as x approaches infinity must approach a single number.

If $\lim_{x \to \infty} f(x) = N$ then y = N is the horizontal asymptote for *f*(x).

To find limits at infinity you will use the following two properties for limits at infinity:

For any positive real number n

1. $\lim_{x \to \infty} \frac{1}{x^n} = 0$
2. $\lim_{x \to -\infty} \frac{1}{x^n} = 0$

The other properties for limits mention earlier (when x approached a) remain the same when you replace "a" with positive or negative infinity.

To find limits at infinity of rational functions you will want to rewrite the function by dividing every term in the numerator and denominator by the largest power of the variable. This will give us a series of fractions with x in the denominator that will approach zero as x approaches infinity. Then you can use the properties for limits at infinity to evaluate the limit.

Let's look at a few examples of finding limits at infinity.

Example 4: Find the following limit at infinity.

$$\lim_{x\to\infty}\frac{2x+5}{x-3}$$

Solution:

First you would divide each term in the numerator and denominator by the largest power of x, which in this case happens to be x.

$$\lim_{x\to\infty}\frac{2x+5}{x-3}=\lim_{x\to\infty}\frac{\frac{2x}{x}+\frac{5}{x}}{\frac{x}{x}-\frac{3}{x}}$$

$$=\lim_{x\to\infty}\frac{2+\frac{5}{x}}{1-\frac{3}{x}}$$

Now you can use property 5 to take the limit of the numerator and denominator.

$$\lim_{x\to\infty}\frac{2+\frac{5}{x}}{1-\frac{3}{x}}=\frac{\lim_{x\to\infty}\left(2+\frac{5}{x}\right)}{\lim_{x\to\infty}\left(1-\frac{3}{x}\right)}$$

Next use property 3 in both the numerator and denominator.

$$\frac{\lim_{x\to\infty}\left(2+\frac{5}{x}\right)}{\lim_{x\to\infty}\left(1-\frac{3}{x}\right)}=\frac{\lim_{x\to\infty}(2)+\lim_{x\to\infty}\left(\frac{5}{x}\right)}{\lim_{x\to\infty}(1)-\lim_{x\to\infty}\left(\frac{3}{x}\right)}$$

Example 4 (Continued):

Finally use property 1 and the property for infinite limits.

$$\frac{\lim_{x\to\infty}(2)+\lim_{x\to\infty}\left(\frac{5}{x}\right)}{\lim_{x\to\infty}(1)-\lim_{x\to\infty}\left(\frac{3}{x}\right)}=\frac{2+0}{1-0}$$

$$=\frac{2}{1}$$

$$=2$$

Therefore $\lim_{x\to\infty}\frac{2x+5}{x-3}=2$

Example 5: Find the following limit at infinity.

$$\lim_{x\to\infty}\frac{x^2+6x+8}{x^3+2x+1}$$

Solution:

Divide all terms by x^3

$$\lim_{x\to\infty}\frac{x^2+6x+8}{x^3+2x+1}=\lim_{x\to\infty}\frac{\frac{x^2}{x^3}+\frac{6x}{x^3}+\frac{8}{x^3}}{\frac{x^3}{x^3}+\frac{2x}{x^3}+\frac{1}{x^3}}$$

Simplify the expression

$$\lim_{x\to\infty}\frac{\frac{x^2}{x^3}+\frac{6x}{x^3}+\frac{8}{x^3}}{\frac{x^3}{x^3}+\frac{2x}{x^3}+\frac{1}{x^3}}=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{6}{x^2}+\frac{8}{x^3}}{1+\frac{2}{x^2}+\frac{1}{x^3}}$$

Use property 5

$$\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{6}{x^2}+\frac{8}{x^3}}{1+\frac{2}{x^2}+\frac{1}{x^3}}=\frac{\lim_{x\to\infty}\left(\frac{1}{x}+\frac{6}{x^2}+\frac{8}{x^3}\right)}{\lim_{x\to\infty}\left(1+\frac{2}{x^2}+\frac{1}{x^3}\right)}$$

Example 5 (Continued):

Use property 3

$$\frac{\lim\limits_{x\to\infty}\left(\frac{1}{x}+\frac{6}{x^2}+\frac{8}{x^3}\right)}{\lim\limits_{x\to\infty}\left(1+\frac{2}{x^2}+\frac{1}{x^3}\right)}=\frac{\lim\limits_{x\to\infty}\left(\frac{1}{x}\right)+\lim\limits_{x\to\infty}\left(\frac{6}{x^2}\right)+\lim\limits_{x\to\infty}\left(\frac{8}{x^3}\right)}{\lim\limits_{x\to\infty}(1)+\lim\limits_{x\to\infty}\left(\frac{2}{x^2}\right)+\lim\limits_{x\to\infty}\left(\frac{1}{x^3}\right)}$$

Use property 1 and the property of infinite limits and simplify.

$$\frac{\lim\limits_{x\to\infty}\left(\frac{1}{x}\right)+\lim\limits_{x\to\infty}\left(\frac{6}{x^2}\right)+\lim\limits_{x\to\infty}\left(\frac{8}{x^3}\right)}{\lim\limits_{x\to\infty}(1)+\lim\limits_{x\to\infty}\left(\frac{2}{x^2}\right)+\lim\limits_{x\to\infty}\left(\frac{1}{x^3}\right)}=\frac{0+0+0}{1+0+0}$$

$$=\frac{0}{1}$$

$$=0$$

Therefore $\lim\limits_{x\to\infty}\frac{x^2+6x+8}{x^3+2x+1}=0$