Antiderivatives

Up until now you have started with a given function and have found its derivative by applying the appropriate derivative rules. The derivatives were used to determine the rate of change of the given function and to determine its extrema (relative and absolute).

However, sometimes you will not know what the function is and all you have to work with is the derivative of the function or the function's rate of change. We can work backwards from the derivative to find the original function by applying the process called antidifferentiation.

Before we start looking at some examples, lets look at the process of find the antiderivative of a function. The first derivative rules you learned dealt with finding the derivative of a constant (C) and a power function (*x* raised to some exponent *n* -- x^n).

The derivative of any constant number (no matter what form it is in) is zero.

$$f(x)=10 \qquad g(x)=\frac{1}{2} \qquad h(x)=\sqrt{13}$$
$$f'(x)=0 \qquad g'(x)=0 \qquad h'(x)=0$$

Therefore, when we are finding the antiderivative of a function we must add the unknown constant of C to the end of the antiderivative. We must use C because without more information about the original function we don't know what was the value of C, which means there are an unlimited number of functions that could have the same derivative.

Look at the following three functions for example. They are all three different but have the same derivative because the only difference between them is the value of the constant in the original function.

$$f(x)=x^2-5 \qquad g(x)=x^2+\pi \qquad h(x)=x^2+\frac{\sqrt{17}}{4}$$
$$f'(x)=2x \qquad g'(x)=2x \qquad h'(x)=2x$$

So our first antiderivative rule is:

Constant Rule

If $f'(x)=0$ then $f(x)=C$

Now lets look at what happens to the power function. A power function is in the form of x^n and its derivative is found by applying the power rule. The power rule tells us we can find the derivative by subtracting 1 from the exponent "n" and then multiplying the function by "n".

If $f(x) = x^n$ then $f'(x) = n \cdot x^{n-1}$

In order to find the antiderivative of a power function we must undo the differentiation process. Since we subtracted 1 from the exponent we will now add 1 back to the exponent to undo the subtraction. We must also undo the multiplication of the exponent in the derivative by now dividing by the exponent of "n+1".

$$f'(x) = x^n$$
$$f(x) = \frac{x^{n+1}}{n+1}$$

Since we are finding the antiderivative we must also add the constant C at the end. So the rule for finding the antiderivative of a power function is:

Power Rule

If $f'(x) = x^n$ then $f(x) = \frac{x^{n+1}}{n+1} + C$

Note: The exponent $n \neq -1$ because if n is -1 then $n + 1$ would be 0 and division by 0 is undefined. You will learn later how to find the antiderivative when $n = -1$.

When you find the antiderivative of a function you will use what is called Integral notation $\int f(x)\,dx$. In this notation $\int$ is called the integral sign and $f(x)$ is called the integrand. $\int f(x)\,dx$ would be read as "the indefinite integral of f(x) with respect to x".

Indefinite Integral

If $F'(x) = f(x)$ then $\int f(x)\,dx = F(x) + C$

Example 1: Find the indefinite integral of $f'(x) = 6$

Solution:

Since the derivative does not contain an "*x*" term we will first insert x^0 into the derivative and then apply the power rule for integration.

$$\begin{aligned} F(x) &= \int f'(x)\,dx \\ &= \int 6dx \\ &= \int 6x^0 dx \\ &= \frac{6x^{0+1}}{0+1} + C \\ &= 6x + C \end{aligned}$$

Example 2: Find the indefinite integral of x^3

Solution:

$$\begin{aligned} F(x) &= \int f'(x)\,dx \\ &= \int x^3 dx \\ &= \frac{x^{3+1}}{3+1} + C \\ &= \frac{x^4}{4} + C \end{aligned}$$

When finding derivatives there were rules for dealing with a constant times a function and the sum or difference of functions. Antiderivatives also have rules for these situations that are very similar to the derivative rules.

Constant Multiple Rule

If you have a constant times a function such as $k\,f(x)$ then the antiderivative would be equal to the constant k times the antiderivative of the function

$$\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx$$

Example 3: Find the antiderivative of $4x^3$.

Solution:

First setup the indefinite integral

$$\int f(x)\,dx = \int 4x^3\,dx$$

Now apply the constant multiple rule

$$\begin{aligned}\int f(x)\,dx &= \int 4x^3\,dx\\ &= 4\int x^3\,dx\end{aligned}$$

Find the antiderivative using the power rule for integration

$$\begin{aligned}\int f(x)\,dx &= \int 4x^3\,dx\\ &= 4\int x^3\,dx\\ &= 4\left[\frac{x^{3+1}}{3+1}\right]+C\\ &= 4\left[\frac{x^4}{4}\right]+C\\ &= x^4+C\end{aligned}$$

The sum or difference rule for derivatives says that when finding the derivative of a function that has more than one term added or subtracted together then you must find the derivative of each individual term and add or subtract their derivatives. Therefore, when finding the antiderivative of a function with multiple terms you must find the antiderivative of each term individually and then add or subtract them.

Sum or Difference Rule

If the derivative is the sum or difference of functions then you will find the indefinite integral of each term.

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example 4: Use the rules of integration discussed so far to find the antiderivative of $x^2 - 4x + 3$.

Solution:

First setup the indefinite integral

$$\int f(x)\,dx = \int \left(x^2 - 4x + 3\right) dx$$

Apply the Sum or Difference rule

$$\begin{aligned}\int f(x)\,dx &= \int \left(x^2 - 4x + 3\right) dx \\ &= \int \left(x^2\right) dx - \int (4x)\,dx + \int (3)\,dx\end{aligned}$$

Apply the Constant Multiple rule

$$\begin{aligned}\int f(x)\,dx &= \int \left(x^2 - 4x + 3\right) dx \\ &= \int \left(x^2\right) dx - \int (4x)\,dx + \int \left(3x^0\right) dx \\ &= \int \left(x^2\right) dx - 4\int (x)\,dx + 3\int \left(x^0\right) dx\end{aligned}$$

Note: The constant 3 can be written as $3x^0$ since x^0 is equal to 1.
$3x^0 = 3(1) = 3$

Apply the Power rule

$$\begin{aligned}\int f(x)\,dx &= \int \left(x^2 - 4x + 3\right) dx \\ &= \int \left(x^2\right) dx - \int (4x)\,dx + \int \left(3x^0\right) dx \\ &= \int \left(x^2\right) dx - 4\int (x)\,dx + 3\int \left(x^0\right) dx \\ &= \left(\frac{x^{2+1}}{2+1}\right) - 4\left(\frac{x^{1+1}}{1+1}\right) + 3\left(\frac{x^{0+1}}{0+1}\right) + C \\ &= \frac{x^3}{3} - 4\left(\frac{x^2}{2}\right) + 3\left(\frac{x}{1}\right) + C \\ &= \frac{x^3}{3} - 2x^2 + 3x + C\end{aligned}$$

So far the examples have dealt with functions that had positive integers as the exponents. There will be times when you must first rewrite the function so that it has either a negative exponent or a fractional exponent before finding the antiderivative.

A few examples of this would be if you have:

1. a fraction with a variable raised to an exponent in the denominator such as $\frac{2}{x^3}$ which would be rewritten by using the properties of exponents as $2x^{-3}$

2. a variable inside a radical such as $\sqrt[3]{x^2}$ which would be rewritten as $x^{2/3}$

3. a polynomial raised to an exponent such as $(x^2 - 2)^2$. In the next section, you will learn a formula that can be applied in these situations but for now you will have multiply (expand) the polynomial before applying any of the antiderivative formulas

Example 5: Find the antiderivative of $2\sqrt[3]{x^4} + \frac{1}{x^2}$.

Solution:

First rewrite the derivative using fractional and/or negative exponents

$$2\sqrt[3]{x^4} + \frac{1}{x^2} = 2x^{4/3} + x^{-2}$$

Setup the indefinite integral

$$\int f(x)\,dx = \int \left(2x^{4/3} + x^{-2}\right) dx$$

Apply the rules of integration

$$\begin{aligned}\int f(x)\,dx &= \int \left(2x^{4/3} + x^{-2}\right) dx \\ &= 2\int \left(x^{4/3}\right) dx + \int \left(x^{-2}\right) dx \\ &= 2\left(\frac{x^{4/3+1}}{4/3+1}\right) + \left(\frac{x^{-2+1}}{-2+1}\right) + C \\ &= 2\left(\frac{x^{4/3+3/3}}{4/3+3/3}\right) + \left(\frac{x^{-1}}{-1}\right) + C \\ &= 2\left(\frac{x^{7/3}}{7/3}\right) - x^{-1} + C\end{aligned}$$

Example 5 (Continued):

$$\int f(x)\,dx = 2\left(\frac{x^{7/3}}{7/3}\right) - x^{-1} + C$$

$$= 2\left(\frac{3}{7}x^{7/3}\right) - \frac{1}{x} + C$$

$$= \frac{6}{7}x^{7/3} - \frac{1}{x} + C$$

Example 6: Find the antiderivative of $(x + 3)^2$.

Solution:

First, expand the polynomial

$$\begin{aligned}(x + 3)^2 &= (x + 3)(x + 3)\\ &= x^2 + 3x + 3x + 9\\ &= x^2 + 6x + 9\end{aligned}$$

Now setup the indefinite integral

$$\int f(x)\,dx = \int \left(x^2 + 6x + 9\right) dx$$

Apply the rules of integration

$$\int f(x)\,dx = \int \left(x^2 + 6x + 9\right) dx$$

$$= \int x^2 dx + \int 6x\,dx + \int 9\,dx$$

$$= \int x^2 dx + 6\int x\,dx + 9\int dx$$

$$= \frac{x^3}{3} + 6\left(\frac{x^2}{2}\right) + 9x + C$$

$$= \frac{1}{3}x^3 + 3x^2 + 9x + C$$

The next rules that we will review are those for finding the antiderivatives of exponential functions, such as e^x or a^x. Back in chapter 4 you learned that the derivative of e^x was simply the original function e^x. However if the base of the exponential function is not e (a^x for example) then the derivative of is equal to the original function times the natural log of its base.

	Derivative rule	Integration rule
Base e	$D_x\left[e^x\right]=e^x$	$\int e^x dx = e^x + C$
Base a	$D_x\left[a^x\right]=a^x \ln a$	$\int a^x dx = \dfrac{a^x}{\ln a} + C$

If the exponent also contains a constant then we must also divide by the constant when finding the antiderivative to undo the multiplication that would occur in the derivative.

	Derivative rule	Integration rule
Base e	$D_x\left[e^{kx}\right]=k\cdot e^{kx}$	$\int e^{kx} dx = \dfrac{e^x}{k} + C$
Base a	$D_x\left[a^{kx}\right]=k\cdot a^{kx} \ln a$	$\int a^{kx} dx = \dfrac{a^{kx}}{k\cdot\ln a} + C$

Example 7: Find the antiderivative of $-3e^{2t}$.

Solution:

Setup the indefinite integral

$$\int f\left(t\right)dt = \int -3e^{2t}dt$$

Apply the rules of integration

$$\begin{aligned}\int f\left(t\right)dt &= \int -3e^{2t}dt\\ &= -3\int e^{2t}dt\\ &= -3\frac{e^{2t}}{2}+C\\ &= -\frac{3}{2}e^{2t}+C\end{aligned}$$

Earlier we mentioned that you couldn't use the power rule of integration in a situation where the exponent is negative one. The antiderivative of x^{-1} is a special rule. As with the other antiderivatives what you want to do is look back at the derivative rules. In this case, x^{-1} could also be written in fraction form as 1/x. In our derivative formulas, the derivative of the natural log of x is equal to 1/x. Therefore, the antiderivative of x^{-1} would be the natural log of x. However, since we cannot take the log of a negative number we must place x within the absolute value sign.

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

Example 8: Find the antiderivative of $8x^{-1}$.

Solution:

Setup the indefinite integral

$$\int f(x)\,dx = \int 8x^{-1}dx$$

Apply the rules of integration

$$\begin{aligned}\int f(x)\,dx &= \int 8x^{-1}dx \\ &= 8\int x^{-1}dx \\ &= 8\ln|x| + C\end{aligned}$$

Up until now we have not been given enough information in order to determine the value of the constant "C" added to the end of the antiderivative. However, in some of the application problems you will be provided with other information that will allow us to find the value of C.

Example 9: Find the cost function if the marginal cost function is $C'(x) = x^2 - 2x + 3$ and the cost of producing 3 units is $15 [or C(3) = 15].

Solution:

We would begin by finding the antiderivative (cost function).

$$\begin{aligned}\int C'(x)\,dx &= \int \left(x^2 - 2x + 3\right) dx \\ &= \int x^2\,dx - \int 2x\,dx + \int 3\,dx \\ &= \int x^2\,dx - 2\int x\,dx + 3\int dx \\ &= \frac{x^3}{3} - 2\left(\frac{x^2}{2}\right) + 3x + C \\ &= \frac{1}{3}x^3 - x^2 + 3x + C\end{aligned}$$

Now use the cost of producing 3 units to find C.

$$\begin{aligned}C(x) &= \frac{1}{3}x^3 - x^2 + 3x + C \\ C(3) &= \frac{1}{3}(3)^3 - (3)^2 + 3(3) + C \\ 15 &= \frac{1}{3}(3)^3 - (3)^2 + 3(3) + C \\ 15 &= \frac{1}{3}(27) - (9) + 9 + C \\ 15 &= 9 + C \\ 6 &= C\end{aligned}$$

Replace C with its calculated value

$$\begin{aligned}C(x) &= \frac{1}{3}x^3 - x^2 + 3x + C \\ C(x) &= \frac{1}{3}x^3 - x^2 + 3x + 6\end{aligned}$$