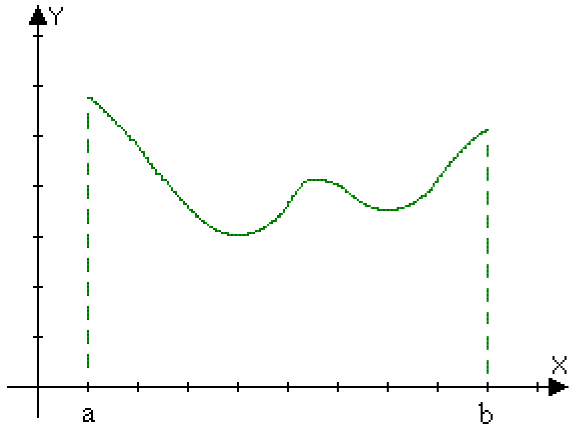
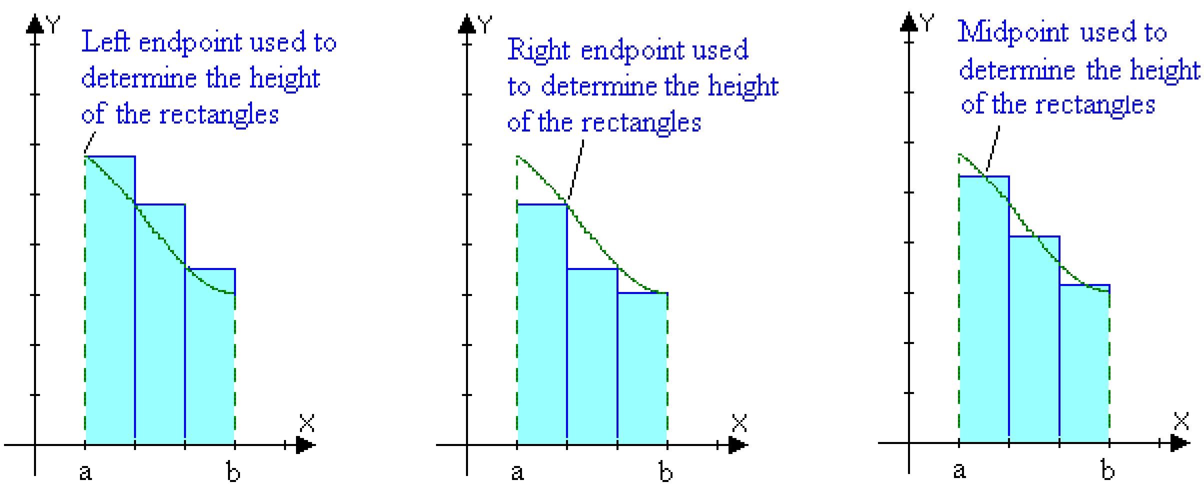
Area and the Definite Integral

In previous courses you learned how to find the area of various geometric shapes (such as squares, triangles, and circles) based on given formulas. However, what if you were asked to find the area under an irregular shaped curve such as:



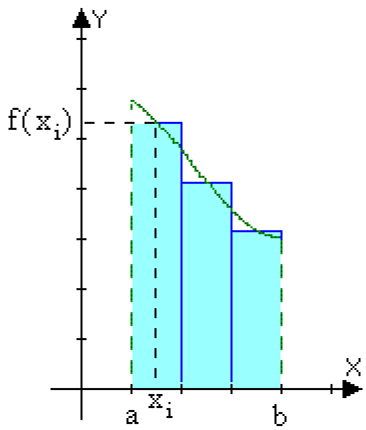
In order to find the area under a curve like this you would use a geometric shape for which you already know the formula. The geometric shape that would make the most sense and give you the best approximation of the area under the curve would be a rectangle. The interval between "a" and "b" would be divided up with rectangles of equal width.

The placement of the rectangles can be done in three different ways: left, right, or midpoint.



As you can see from the graphs, using the left endpoint to determine the height of the graph will result in finding an area greater than that which is under the curve. If you use the right endpoint to determine the height the resulting area will be less than that which is under the curve.

Therefore, the best choice is to use the midpoint. If you use the midpoint then the parts of the rectangle to extend beyond the curve will basically balance out the parts omitted under the graph. If we let x_i represent the midpoint of the intervals then $f(x_i)$ would be the height of the rectangles.



Now that we know what the height of each rectangle is equal to we need to determine what will be the width of each rectangle. We know that the width of each interval (or rectangle) is to be equal so to determine this width we would take the difference between the endpoints "a" and "b" and divide it by the number of intervals "n" that will be used.

$$width = \frac{b-a}{n} = \Delta x$$

We just saw how using the midpoint for the height of the rectangles would improve the accuracy of the area approximation. We can also get a better approximation by increasing the number of rectangles (intervals). This will cause the width of each rectangle to approach zero as the number of rectangles approaches infinity.

The area of each rectangle would equal $f\left(x_i\right)\cdot\Delta x$. Therefore, the area under the curve approximated by using n rectangles would be equal to the summation of the areas of all the rectangles.

$$\text{Area of all rectangles} = \sum_{i=1}^{n} f\left(x_i\right)\cdot\Delta x$$

Example 1: Approximate the area under the graph f(x) = 16 – x^2 and above the x-axis from x = 0 to x = 4 using four equal intervals. Use the midpoint of each interval to determine the height of the rectangles.

Solution:

Step 1: Determine the width of each interval

$$\begin{aligned}\Delta x &= \frac{b-a}{n}\\ &= \frac{4-0}{4}\\ &= 1\end{aligned}$$

Step 2: Determine the midpoints for each interval

$$\begin{aligned}x_1 &= \frac{0+1}{2} = 0.5\\ x_2 &= \frac{1+2}{2} = 1.5\\ x_3 &= \frac{2+3}{2} = 2.5\\ x_4 &= \frac{3+4}{2} = 3.5\end{aligned}$$

Step 3: Find the approximate area under the curve

$$\begin{aligned}\sum_{i=1}^{4} f(x_i)\cdot\Delta x &= f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x + f(x_4)\cdot\Delta x\\ &= f(x_1)\cdot(1) + f(x_2)\cdot(1) + f(x_3)\cdot(1) + f(x_4)\cdot(1)\\ &= f(x_1) + f(x_2) + f(x_3) + f(x_4)\\ &= \left[16-(0.5)^2\right] + \left[16-(1.5)^2\right] + \left[16-(2.5)^2\right] + \left[16-(3.5)^2\right]\\ &= (16-0.25) + (16-2.25) + (16-6.25) + (16-12.25)\\ &= 15.75 + 13.75 + 9.75 + 3.75\\ &= 43\end{aligned}$$

Example 2: Using the same function and endpoints repeat example 1 but this time use 8 equal intervals.

Solution:

Step 1: Determine the width of each interval

$$\begin{aligned}\Delta x &= \frac{b-a}{n}\\ &= \frac{4-0}{8}\\ &= \frac{1}{2}\\ &= 0.5\end{aligned}$$

Step 2: Determine the midpoints for each interval

$$x_1 = \frac{0+0.5}{2} = 0.25 \qquad x_5 = \frac{2+2.5}{2} = 2.25$$

$$x_2 = \frac{0.5+1}{2} = 0.75 \qquad x_6 = \frac{2.5+3}{2} = 2.75$$

$$x_3 = \frac{1+1.5}{2} = 1.25 \qquad x_7 = \frac{3+3.5}{2} = 3.25$$

$$x_4 = \frac{1.5+2}{2} = 1.75 \qquad x_8 = \frac{3.5+4}{2} = 3.75$$

Step 3: Find the approximate area under the curve

$$\begin{aligned}\sum_{i=1}^{8} f(x_i)\cdot\Delta x &= f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x + f(x_4)\cdot\Delta x + f(x_5)\cdot\Delta x +\\ &\quad f(x_6)\cdot\Delta x + f(x_7)\cdot\Delta x + f(x_8)\cdot\Delta x\end{aligned}$$

$$\begin{aligned}&= f(x_1)\cdot(0.5) + f(x_2)\cdot(0.5) + f(x_3)\cdot(0.5) + f(x_4)\cdot(0.5) +\\ &\quad f(x_5)\cdot(0.5) + f(x_6)\cdot(0.5) + f(x_7)\cdot(0.5) + f(x_8)\cdot(0.5)\end{aligned}$$

$$\begin{aligned}&= [f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7) +\\ &\quad f(x_8)](0.5)\end{aligned}$$

Example 2 (Continued):

$$\sum_{i=1}^{8} f\left(x_i\right)\cdot \Delta x = [\,f\left(x_1\right)+f\left(x_2\right)+f\left(x_3\right)+f\left(x_4\right)+f\left(x_5\right)+f\left(x_6\right)+f\left(x_7\right)+ f\left(x_8\right)]\left(0.5\right)$$

$$= [\,f\left(0.25\right)+f\left(0.75\right)+f\left(1.25\right)+f\left(1.75\right)+f\left(2.25\right)+f\left(2.75\right) + f\left(3.25\right)+f\left(3.75\right)]\left(0.5\right)$$

$$= [\left(16-0.25^2\right)+\left(16-0.75^2\right)+\left(16-1.25^2\right)+\left(16-1.75^2\right)+ \left(16-2.25^2\right)+\left(16-2.75^2\right)+\left(16-3.25^2\right)+\left(16-3.75^2\right)]\left(0.5\right)$$

$$= [\left(16-0.0625\right)+\left(16-0.5625\right)+\left(16-1.5625\right)+\left(16-3.0625\right) +\left(16-5.0625\right)+\left(16-7.5625\right)+\left(16-10.5625\right)+\left(16-14.0625\right)]\left(0.5\right)$$

$$= (15.9375+15.4375+14.4375+12.9375+10.9375+8.4375+ 5.4375+1.9375)\left(0.5\right)$$

$$= \left(85.5\right)\left(0.5\right)$$
$$= 42.75$$

If the number of rectangles is allowed to approach infinity then we could find the exact area under the curve by taking the limit as n approaches infinity of the previous formula.

$$\text{Exact area} = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i\right)\cdot \Delta x$$

This limit is called the definite integral of f(x) from a to b and is written as:

$$\int_a^b f\left(x\right)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i\right)\cdot \Delta x$$

You will see how this integral is used to calculate the area in the next section.