Evaluating Definite Integrals

There are a few properties that you should remember in order to assist you in evaluating definite integrals.

$$\int_a^a f(x)\,dx = 0$$

$\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$; where k is any real constant

$$\int_a^b \left[f(x) \pm g(x) \right] dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$; where c is any real number in the interval $[a, b]$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Example 1: Evaluate the definite integral $\int_0^3 \left(x^2 - 3x + 2\right) dx$

Solution:

Step 1: Apply the sum/difference integral property

$$\int_0^3 \left(x^2 - 3x + 2\right) dx = \int_0^3 \left(x^2\right) dx - \int_0^3 (3x)\,dx + \int_0^3 (2)\,dx$$

Step 2: Apply the constant property

$$\int_0^3 \left(x^2 - 3x + 2\right) dx = \int_0^3 \left(x^2\right) dx - 3 \cdot \int_0^3 (x)\,dx + 2 \cdot \int_0^3 dx$$

Example 1 (Continued):

Step 3: Find the antiderivatives of each term

$$\int_0^3 \left(x^2 - 3x + 2\right)dx = \left(\frac{x^3}{3}\right)_0^3 - 3\left(\frac{x^2}{2}\right)_0^3 + 2\left(x\right)_0^3$$

$$= \frac{1}{3}\left(x^3\right)_0^3 - \frac{3}{2}\left(x^2\right)_0^3 + 2\left(x\right)_0^3$$

Step 4: Evaluate

$$\int_0^3 \left(x^2 - 3x + 2\right)dx = \frac{1}{3}\left(x^3\right)_0^3 - \frac{3}{2}\left(x^2\right)_0^3 + 2\left(x\right)_0^3$$

$$= \frac{1}{3}\left(3^3 - 0^3\right) - \frac{3}{2}\left(3^2 - 0^2\right) + 2\left(3 - 0\right)$$

$$= \frac{1}{3}\left(27\right) - \frac{3}{2}\left(9\right) + 2\left(3\right)$$

$$= 9 - \frac{27}{2} + 6$$

$$= 15 - \frac{27}{2}$$

$$= \frac{30}{2} - \frac{27}{2}$$

$$= \frac{3}{2}$$

As with the indefinite integrals, integration by substitution can be perform on definite integrals. The main thing to remember when using substitution with definite integrals is that the upper and lower limits of the integral must be changed to match the new variable "*u*".

Example 2: Evaluate the definite integral $\int_0^1 2x\sqrt{5x^2 + 4}\ dx$

Solution:

Step 1: Identify the inside function and set it equal to *u*.

In this example, the inside function will be the radicand of $5x^2 + 4$.

So, $u = 5x^2 + 4$.

Example 2 (Continued):

Step 2: Find the differential of *u*.

$$u = 5x^2 + 4$$
$$\frac{du}{dx} = 10x$$
$$du = 10x \ \ dx$$

Step 3: Check if the differential of *u* is present in the integrand

du = 10x dx but the integrand only contains 2x dx so you must divide both sides by 5 in order to perform the substitution.

$$du = 10x \ \ dx$$
$$\frac{du}{5} = \frac{10x}{5} \ \ dx$$
$$\frac{1}{5}du = 2x \ \ dx$$

Step 4: Adjust the limits of the integral to match with the new variable of u.

If x = 1 then

$u = 5x^2 + 4$
$u = 5(1)^2 + 4$
$u = 9$

If x = 0 then

$u = 5x^2 + 4$
$u = 5(0)^2 + 4$
$u = 4$

Step 5: Perform the substitution and change the integral limits

$$\int_0^1 2x\sqrt{5x^2+4}\ dx = \int_0^1 \sqrt{5x^2+4}\ 2x\, dx$$
$$= \int_4^9 \sqrt{u}\ \frac{1}{5}\, du$$

Example 2 (Continued):

Step 6: Evaluate the integral

$$\int_4^9 \sqrt{u}\ \frac{1}{5}\,du = \frac{1}{5}\int_4^9 (u)^{\frac{1}{2}}\ \ du$$

$$= \frac{1}{5}\left(\frac{u^{\frac{3}{2}}}{3/2}\right)\Bigg|_4^9$$

$$= \frac{1}{5}\left(\frac{2}{3}\right)\left(u^{\frac{3}{2}}\right)\Bigg|_4^9$$

$$= \frac{2}{15}\left(9^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$$

$$= \frac{2}{15}\left(3^3 - 2^3\right)$$

$$= \frac{2}{15}(27-8)$$

$$= \frac{2}{15}(19)$$

$$= \frac{38}{15}$$

In some problems where you are asked to find the area of a region between the x-axis and a function over an interval [a, b] the entire region or some portion of it may be below the x-axis. The area of the region below the x-axis using the Fundamental Theorem would be negative. Therefore, you must be careful when dealing with problems of this nature since area is suppose to be nonnegative.

The problem with negative areas can be avoided by:

1. taking the absolute value of the integral, or

$$\int_a^b f(x)\,dx = \left|\int_a^b f(x)\,dx\right|$$

2. applying the rules of definite integrals to reverse the upper and lower limits

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

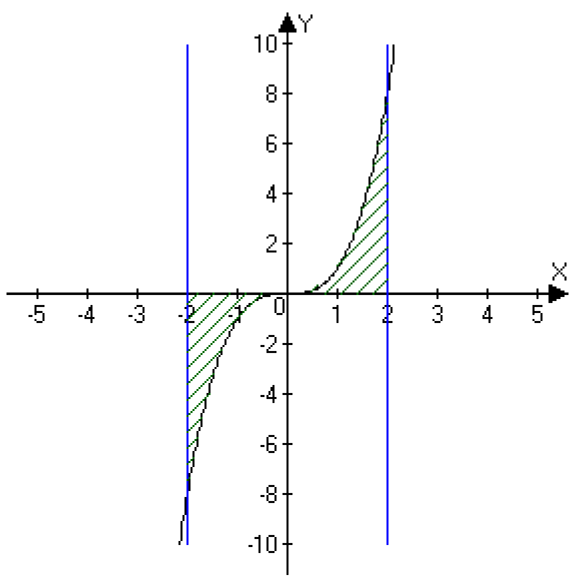
Below are some guidelines to follow when asked to find the area of a bounded region:

1. Sketch a graph of the problem
2. Locate any *x*-intercepts in the given interval [a, b]
3. Identify any regions that are below the *x*-axis
4. If there are regions above and below the *x*-axis split up the interval and use separate integrals to evaluate each subregion.
5. Add the separate areas together to find the total area.

Example 3: Find the area between the *x*-axis and the function $f(x) = x^3$ over the interval [-2, 2].

Solution:

Step 1: Sketch a graph of the problem



Example 3 (Continued):

Step 2: Locate any *x*-intercepts in the given interval [a, b]

$$f(x) = x^3$$
$$0 = x^3$$
$$0 = x$$

There is an x-intercept at x = 0, which can be seen in the graph.

Step 3: Identify any regions that are below the *x*-axis

The region over the interval [-2, 0] is below the x-axis.

Step 4: Split up the interval and use separate integrals to evaluate each subregion

$$\begin{aligned} Area &= \int_{-2}^{2} \left(x^3\right)\, dx \\ &= \int_{-2}^{0} \left(x^3\right)\, dx + \int_{0}^{2} \left(x^3\right)\, dx \end{aligned}$$

Now since the interval [-2, 0] is below the x-axis we must adjust the integral for this interval. Therefore we will take the absolute value of this integral to assure that the value will be positive.

$$\begin{aligned} \int_{-2}^{0} \left(x^3\right)\, dx + \int_{0}^{2} \left(x^3\right)\, dx &= \left|\int_{-2}^{0} \left(x^3\right)\, dx\right| + \int_{0}^{2} \left(x^3\right)\, dx \\ &= \left|\left(\frac{x^4}{4}\right)\Bigg|_{-2}^{0}\right| + \left(\frac{x^4}{4}\right)\Bigg|_{0}^{2} \\ &= \left|\frac{0^4}{4} - \frac{(-2)^4}{4}\right| + \left(\frac{2^4}{4} - \frac{0^4}{4}\right) \\ &= |0-4| + (4-0) \\ &= 4+4 \\ &= 8 \end{aligned}$$

The area between the x-axis and f(x) over the interval [-2, 2] would be 8.

Let's look at what would happen if we did not separate the interval at the x-intercept of zero.

$$\begin{aligned} Area &= \int_{-2}^{2} \left(x^3\right)\, dx \\ &= \left(\frac{x^4}{4}\right)\Bigg|_{-2}^{2} \\ &= \left(\frac{2^4}{4} - \frac{(-2)^4}{4}\right) \\ &= (4-4) \\ &= 0 \end{aligned}$$

Looking at the graph you can obvious tell that the area cannot be equal to zero, but since the intervals were not separated the area for the interval [-2, 0] offset the area for the interval [0, 2].

The average value of a function over the interval [*a*, *b*] is determined by multiplying the definite integral by the fraction of 1 over (*b* – *a*).

$$Average\ \ value = \frac{1}{b-a}\int_a^b f\left(x\right)dx$$

Example 4: Find the average value of $f(x) = x^2 + 3$ on the interval [2, 4].

Solution:

$$\begin{aligned} Average\ \ value &= \frac{1}{b-a}\int_a^b f\left(x\right)dx \\ &= \frac{1}{4-2}\int_2^4 \left(x^2+3\right)dx \\ &= \frac{1}{2}\left(\frac{x^3}{3}+3x\right)\Bigg|_2^4 \\ &= \frac{1}{2}\left[\left(\frac{4^3}{3}+3(4)\right)-\left(\frac{2^3}{3}+3(2)\right)\right] \end{aligned}$$

Example 4 (Continued):

$$\begin{aligned} Average\ \ value &= \frac{1}{2}\left[\frac{64}{3}+12-\frac{8}{3}-6\right] \\ &= \frac{1}{2}\left(\frac{56}{3}+6\right) \\ &= \frac{28}{3}+3 \\ &= \frac{37}{3} \end{aligned}$$