The Fundamental Theorem of Calculus

In the previous section, the definite integral was defined as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \quad \text{where } \Delta x = \frac{b-a}{n}$$

If you were to use this formal definition to evaluate the definite integral then you would have to divide the interval [a, b] into an infinite number of rectangles (n = ∞) and then sum their areas. The Fundamental Theorem of Calculus provides us an easier and faster way to evaluate a definite integral without having to use the limit of a sum.

If f is continuous on the closed interval $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

The definition says that F can be <u>any</u> antiderivative of f because the constant C (which is normally added to the end of the antiderivative) will not affect the value of the definite integral.

$$\begin{aligned}\int_a^b f(x)\,dx &= \left[F(x)+C\right]\Big|_a^b \\ &= \left[F(b)+C\right]-\left[F(a)+C\right] \\ &= F(b)+C-F(a)-C \\ &= F(b)-F(a)\end{aligned}$$

As you can see if you were to leave the constant C in the antiderivative it would just end up being subtracted from itself leaving you with $F(b) - F(a)$ just as in the theorem.

Example 1: Evaluate the definite integral $\int_1^2 \left(3x^2\right) dx$.

Solution:

Step 1: Find the antiderivative

$$\begin{aligned} \int \left(3x^2\right) dx &= \frac{3x^3}{3} + C \\ &= x^3 + C \end{aligned}$$

Step 2: Evaluate the definite integral.

$$\begin{aligned} \int_1^2 \left(3x^2\right) dx &= \left(x^3\right)\Big|_1^2 \\ &= \left(2^3\right) - \left(1^3\right) \\ &= 8 - 1 \\ &= 7 \end{aligned}$$

Example 2: Evaluate the definite integral $\int_1^8 2\sqrt[3]{x^4}\, dx$.

Solution:

Step 1: Rewrite the function using fractional exponents

$$2\sqrt[3]{x^4} = 2x^{4/3}$$

Step 2: Find the antiderivative

$$\begin{aligned} \int 2x^{4/3}\, dx &= \frac{2x^{4/3+1}}{4/3+1} + C \\ &= \frac{2x^{7/3}}{7/3} + C \\ &= \frac{3}{7}\left(2x^{7/3}\right) + C \\ &= \frac{6}{7}x^{7/3} + C \end{aligned}$$

Example 2 (Continued):

Step 3: Evaluate the definite integral

$$\int_1^8 2x^{4/3}\,dx = \left(\frac{6}{7}x^{7/3}\right)\Big|_1^8$$
$$= \frac{6}{7}\left(x^{7/3}\right)\Big|_1^8$$
$$= \frac{6}{7}\left((8)^{7/3} - (1)^{7/3}\right)$$
$$= \frac{6}{7}\left(2^7 - 1^7\right)$$
$$= \frac{6}{7}(128 - 1)$$
$$= \frac{6}{7}(127)$$
$$= \frac{762}{7}$$

Example 3: Evaluate the definite integral of $-3e^{2t}$ on the closed interval [0, 4].

Solution:

Step 1: Find the antiderivative

$$\int -3e^{2t}dt = -3\int e^{2t}dt$$
$$= -3\left(\frac{e^{2t}}{2}\right) + C$$
$$= -\frac{3}{2}e^{2t} + C$$

Example 3 (Continued):

Step 2: Evaluate the definite integral

$$\int_0^4 -3e^{2t}\,dx = \left(-\frac{3}{2}e^{2t}\right)\Big|_0^4$$

$$= -\frac{3}{2}\left(e^{2t}\right)\Big|_0^4$$

$$= -\frac{3}{2}\left(e^{2(4)} - e^{2(0)}\right)$$

$$= -\frac{3}{2}\left(e^{8} - e^{0}\right)$$

$$= -\frac{3}{2}\left(e^{8} - 1\right)$$

$$= -\frac{3}{2}e^{8} + \frac{3}{2}$$