Review Exercise Set 21

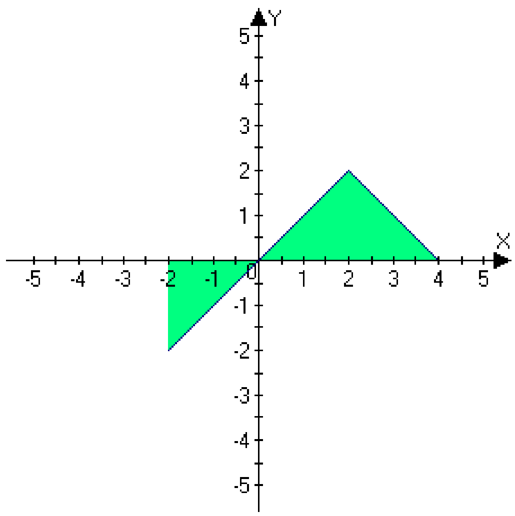
Exercise 1: Approximate the area under the graph of f(x) and above the x-axis by using the midpoints of 4 equal intervals.

$f(x) = 3x - 5$ from $x = -2$ to $x = 2$

Exercise 2: Approximate the area under the graph of g(t) and above the x-axis by using the midpoints of 5 equal intervals.

$g(t) = t^2 - 4t + 5$ from $x = -1$ to $x = 4$

Exercise 3: Find $\int_{-2}^{4} f(x)\,dx$ for the graph of y = f(x) using geometric formulas for the shapes created between the function and the x-axis. Estimate the area using the midpoints of 12 equal intervals. Compare the two areas.



Exercise 4: Find the exact value of the definite integral by using geometric formulas for the shapes formed by the area under the graph of function.

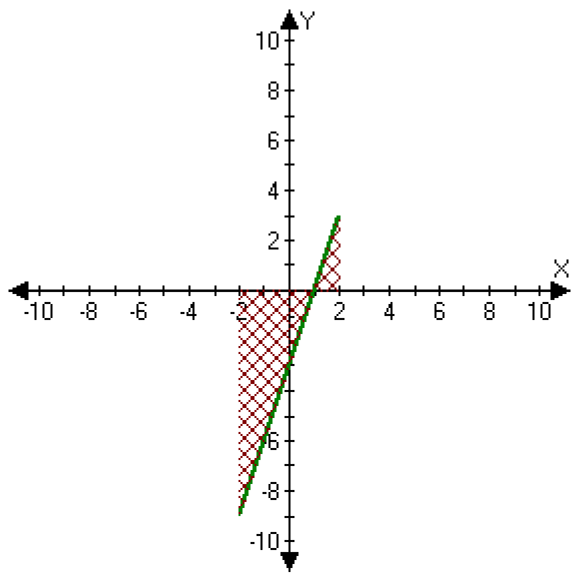
$$\int_{0}^{4} \sqrt{16-x^2}\,dx$$

Review Exercise Set 21 Answer Key

Exercise 1: Approximate the area under the graph of f(x) and above the x-axis by using the midpoints of 4 equal intervals.

$f(x) = 3x - 3$ from $x = -2$ to $x = 2$

Sketch graph of function on the interval [-2, 2]



Determine the intervals

interval width $\Delta x = \frac{b-a}{n} = \frac{2-(-2)}{4} = 1$

intervals: [-2, -1], [-1, 0], [0, 1], and [1, 2]

Determine the midpoints

Since the interval width is 1 and midpoint can be found by adding $\frac{1}{2}$ to the beginning of each interval

For interval [-2, -1], midpoint (x_1) = $-2 + \frac{1}{2} = -\frac{3}{2}$

For interval [-1, 0], midpoint (x_2) = $-1 + \frac{1}{2} = -\frac{1}{2}$

For interval [0, 1], midpoint (x_3) = $0 + \frac{1}{2} = \frac{1}{2}$

For interval [1, 2], midpoint (x_4) = $1 + \frac{1}{2} = \frac{3}{2}$

Exercise 1 (Continued):

Approximate the area

$$\int_{-2}^{2} (3x-3)\,dx$$

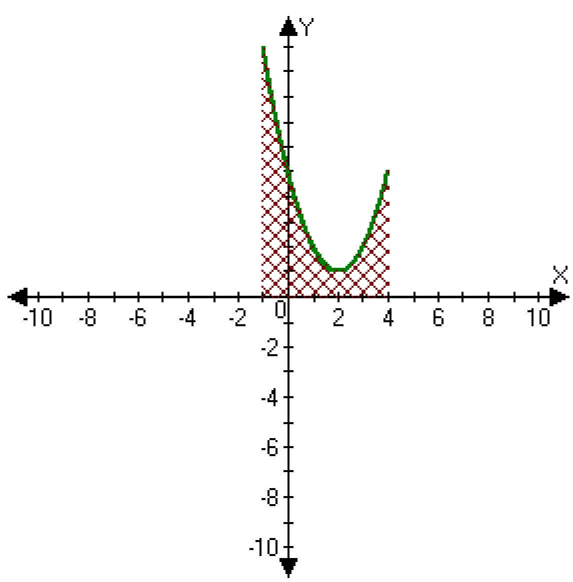
Since the function is below the x-axis on the interval [-2, 1] we must separate the integral into two regions. The region below the x-axis will be multiplied by -1 to ensure the area is a positive value.

$$\begin{aligned}\int_{-2}^{2} (3x-3)\,dx &= -\int_{-2}^{1} (3x-3)\,dx + \int_{1}^{2} (3x-3)\,dx \\ &= -\left[f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \right] + f(x_4)\Delta x \\ &= -f\left(-\frac{3}{2}\right)(1) - f\left(-\frac{1}{2}\right)(1) - f\left(\frac{1}{2}\right)(1) + f\left(\frac{3}{2}\right)(1) \\ &= -\left[3\left(-\frac{3}{2}\right)-3\right] - \left[3\left(-\frac{1}{2}\right)-3\right] - \left[3\left(\frac{1}{2}\right)-3\right] + \left[3\left(\frac{3}{2}\right)-3\right] \\ &= \frac{9}{2}+3+\frac{3}{2}+3-\frac{3}{2}+3+\frac{9}{2}-3 \\ &= 15\end{aligned}$$

Exercise 2: Approximate the area under the graph of g(t) and above the x-axis by using the midpoints of 5 equal intervals.

$g(t) = t^2 - 4t + 5$ from x = -1 to x = 4

Sketch graph of function on the interval [-1, 4]



Exercise 2 (Continued):

Determine the intervals

interval width $\Delta x = \frac{b-a}{n} = \frac{4-(-1)}{5} = 1$

intervals: [-1, 0], [0, 1], [1, 2], [2, 3], and [3, 4]

Determine the midpoints

Since the interval width is 1 and midpoint can be found by adding $\frac{1}{2}$ to the beginning of each interval

For interval [-1, 0], midpoint (x_1) = -1 + $\frac{1}{2}$ = -$\frac{1}{2}$

For interval [0, 1], midpoint (x_2) = 0 + $\frac{1}{2}$ = $\frac{1}{2}$

For interval [1, 2], midpoint (x_3) = 1 + $\frac{1}{2}$ = $\frac{3}{2}$

For interval [2, 3], midpoint (x_4) = 2 + $\frac{1}{2}$ = $\frac{5}{2}$

For interval [3, 4], midpoint (x_5) = 3 + $\frac{1}{2}$ = $\frac{7}{2}$

Approximate the area

$$\int_{-1}^{4}\left(t^2-4t+5\right)dx = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x$$

$$= f\left(-\frac{1}{2}\right)(1) + f\left(\frac{1}{2}\right)(1) + f\left(\frac{3}{2}\right)(1) + f\left(\frac{5}{2}\right)(1) + f\left(\frac{7}{2}\right)(1)$$

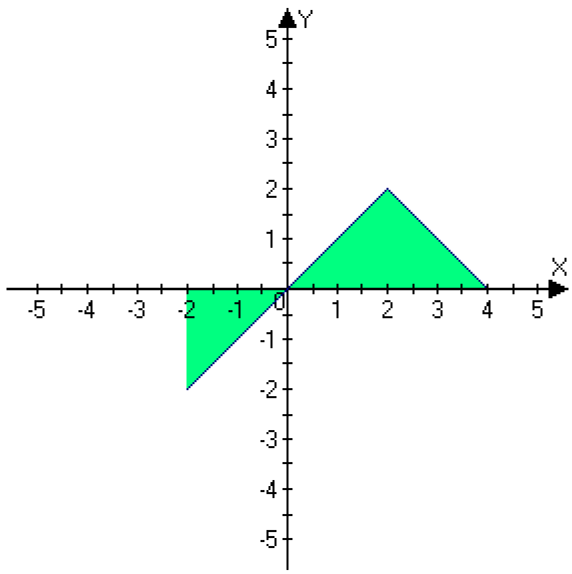
$$= \left[\left(-\frac{1}{2}\right)^2 - 4\left(-\frac{1}{2}\right)+5\right] + \left[\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right)+5\right] +$$

$$\left[\left(\frac{3}{2}\right)^2 - 4\left(\frac{3}{2}\right)+5\right] + \left[\left(\frac{5}{2}\right)^2 - 4\left(\frac{5}{2}\right)+5\right] + \left[\left(\frac{7}{2}\right)^2 - 4\left(\frac{7}{2}\right)+5\right]$$

$$= \frac{1}{4}+2+5+\frac{1}{4}-2+5+\frac{9}{4}-6+5+\frac{25}{4}-10+5+\frac{49}{4}-14+5$$

$$= 16.25$$

Exercise 3: Find $\int_{-2}^{4} f(x)\,dx$ for the graph of y = f(x) using geometric formulas for the shapes created between the function and the x-axis. Estimate the area using the midpoints of 12 equal intervals. Compare the two areas.



Area of triangle above the x-axis

$$A = \frac{1}{2} bh$$

$$A = \frac{1}{2} (4)(2)$$

$$A = 4$$

Area of triangle below the x-axis

$$A = \frac{1}{2} bh$$

$$A = \frac{1}{2} (2)(2)$$

$$A = 2$$

Add the two areas together to find to the total area

Total shaded area = 4 + 2 = 6

Exercise 3 (Continued):

Estimate the area

interval width $\Delta x = \frac{b-a}{n} = \frac{4-(-2)}{12} = \frac{1}{2}$

first midpoint: -2 + $\frac{1}{4}$ = -$\frac{7}{4}$

other midpoints will increase in $\frac{1}{2}$ increments:

$-\frac{5}{4}, -\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}, \frac{13}{4}, \frac{15}{4}$

$$\int_{-2}^{4} f(x)\,dx = -\int_{-2}^{0} f(x)\,dx + \int_{0}^{4} f(x)\,dx$$

$$= -\left[f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \right] + \left[f(x_5)\Delta x + \right.$$
$$f(x_6)\Delta x + f(x_7)\Delta x + f(x_8)\Delta x + f(x_9)\Delta x + f(x_{10})\Delta x + f(x_{11})\Delta x$$
$$\left. + f(x_{12})\Delta x \right]$$
$$= -f\left(-\frac{7}{4}\right)\left(\frac{1}{2}\right) - f\left(-\frac{5}{4}\right)\left(\frac{1}{2}\right) - f\left(-\frac{3}{4}\right)\left(\frac{1}{2}\right) - f\left(-\frac{1}{4}\right)\left(\frac{1}{2}\right) +$$
$$f\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\left(\frac{1}{2}\right) + f\left(\frac{5}{4}\right)\left(\frac{1}{2}\right) + f\left(\frac{7}{4}\right)\left(\frac{1}{2}\right) + f\left(\frac{9}{4}\right)\left(\frac{1}{2}\right) +$$
$$+ f\left(\frac{11}{4}\right)\left(\frac{1}{2}\right) + f\left(\frac{13}{4}\right)\left(\frac{1}{2}\right) + f\left(\frac{15}{4}\right)\left(\frac{1}{2}\right)$$

From the graph we can estimate the values of f(x_n)

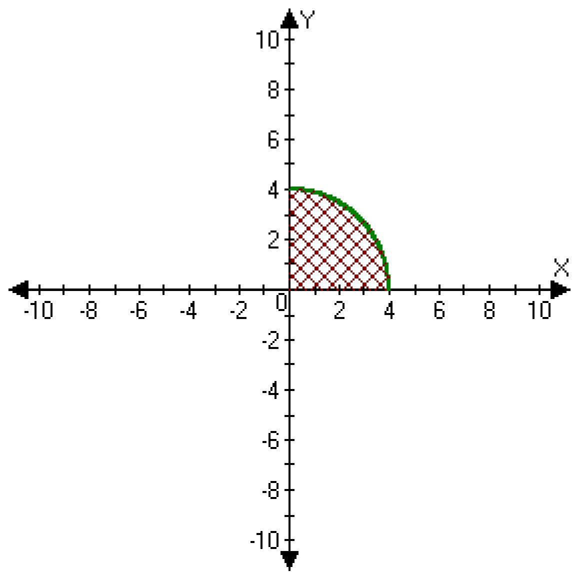
$$\int_{-2}^{4} f(x)\,dx = \frac{7}{8} + \frac{5}{8} + \frac{3}{8} + \frac{1}{8} + \frac{1}{8} + \frac{3}{8} + \frac{5}{8} + \frac{7}{8} + \frac{7}{8} + \frac{5}{8} + \frac{3}{8} + \frac{1}{8}$$
$$= 6$$

The area found under both methods is exactly the same.

Exercise 4: Find the exact value of the definite integral by using geometric formulas for the shapes formed by the area under the graph of function.

$$\int_0^4 \sqrt{16 - x^2}\,dx$$

Sketch graph of function on the interval [0, 4]



Determine the formula to use based on the geometric shape formed

The shaded area is a quarter of a circle, therefore we would take the formula for the area of a circle and multiply it by one-fourth.

$$A = \frac{1}{4}\pi r^2$$

The radius of the circle would be the length of the interval 4 - 0 = 4.

Find the area

$$\begin{aligned} A &= \frac{1}{4}\pi(4)^2 \\ &= \frac{1}{4}\pi(16) \\ &= 4\pi \end{aligned}$$