Law of Sines

So far we have been using the trigonometric functions to solve right triangles. However, what happens when the triangle does not have a right angle? When solving oblique triangles we cannot use the formulas defined for right triangles and must use new ones. For this section, the Law of Sines will be examined in how it can be used to solve oblique triangles.

Definition of the Law of Sines:

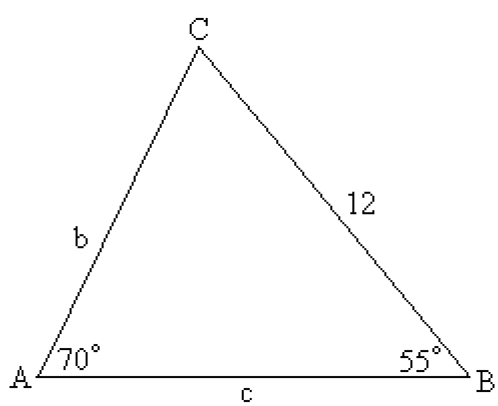
> If A, B, and C are the measurements of the angles of an oblique triangle, and a, b, and c are the lengths of the sides opposite of the corresponding angles, then the ratios of the a side's length to the sine of the angle opposite the side must all be the same.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Applying the Law of Sines:

The Law of Sines can be used to solve for the missing lengths or angle measurements in an oblique triangle as long as two of the angles and one of the sides are known. There are two cases that can exist for this situation. The known side could be the side between the two known angles (Angle-Side-Angle, ASA) or it could be one of the other two sides (Side-Angle-Angle, SAA). Let's now look at a couple of examples of these two situations and how the Law of Sines is used to solve the triangles.

Example 1: Solve the given triangle using the Law of Sines. Round lengths to the nearest tenth and angle measurements to the nearest degree.



A = 70°, B = 55°, and a = 12

Example 1 (Continued):

Solution:

Find angle C

The sum of the angles of a triangle must equal 180° so C would be the difference between 180° and the sum of the other two angles.

$A + B + C = 180°$
$70° + 55° + C = 180°$
$C = 180° - 125°$
$C = 55°$

Find side b

Since we are given angle A and side a we will use these in our ratio setup to find side b.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{12}{\sin 70°} = \frac{b}{\sin 55°}$$
$$\frac{12 \sin 55°}{\sin 70°} = b$$
$b \approx 10.5$

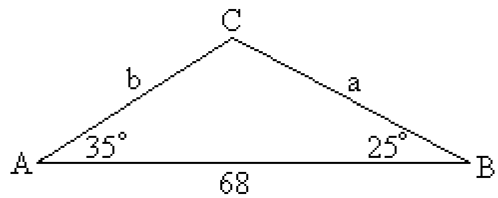
Find side c

Even though we just found side b you will still want to use the measurement that were given to us in the problem. The length of side b is an approximated value and will not produce as good of an answer as the known values for side a and angle A.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{12}{\sin 70°} = \frac{c}{\sin 55°}$$
$$\frac{12 \sin 55°}{\sin 70°} = c$$
$c \approx 10.5$

Since angles B and C are the same the lengths of sides b and c will also be the same.

Example 2: Solve the given triangle using the Law of Sines. Round lengths to the nearest tenth and angle measurements to the nearest degree.



A = 35°, B = 25°, and c = 68

Solution:

Find angle C

A + B + C = 180°
35° + 25° + C = 180°
C = 180° – 60°
C = 120°

Find side b

Since we know the exact measurements for side c and angle C we will use these in our ratio setup to find side b.

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$
$$\frac{68}{\sin 120°} = \frac{b}{\sin 25°}$$
$$\frac{68 \sin 25°}{\sin 120°} = b$$

b ≈ 33.2

Find side a

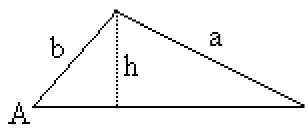
$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{a}{\sin 35°} = \frac{68}{\sin 120°}$$
$$a = \frac{68 \sin 35°}{\sin 120°}$$

a ≈ 45.0

In the two previous cases, ASA and SAA, the Law of Sines can be used to solve the triangles without any problems. However, there is a third case where the Law of Sines might be able to be used depending on the height of the triangle. This last case is called the Ambiguous Case because in some situations it is not possible to use the Law of Sines to solve the triangle or you may get more than one answer. This case is where two of the lengths of the sides are known and one of the angle's measurements is known but not the angle made up by the two sides (Side-Side-Angle, SSA).

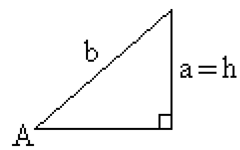
Possible SSA situations ($h = b \sin A$)

One Oblique Triangle



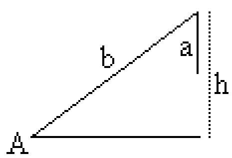
$a > b$ and $a > h$

One Right Triangle



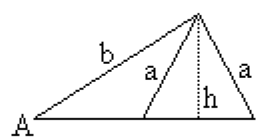
$a = h$

No Triangle



$a < b$ and $a < h$

Two Oblique Triangles



$a < b$ but $a > h$

Example 3: Determine whether the given measurements produce one triangle, two triangles, or no triangle. If a triangle is formed then solve the triangle (or triangles) rounding the sides to the nearest tenth and the angles to the nearest degree.

$$b = 100 \text{ m},\ a = 60 \text{ m, and } \angle A = 28°$$

Solution:

Find the height (h) of the triangle

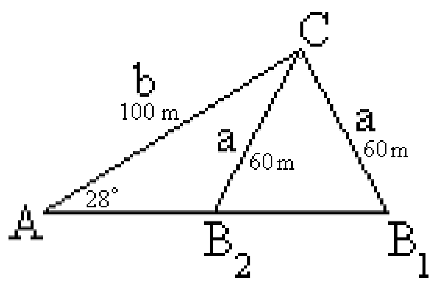
$h = b \sin A$
$h = 100 \sin 28°$
$h \approx 46.9$ m

Example 3 (Continued):

Identify the case that applies (1 triangle, 2 triangles, or no triangle)

Compare sides a and b: 60 m < 100 m
Compare side a and h: 60 m > 46.9 m

Since side a is greater than the height but less than side b there will be 2 possible triangles formed.



Find B_1 and B_2

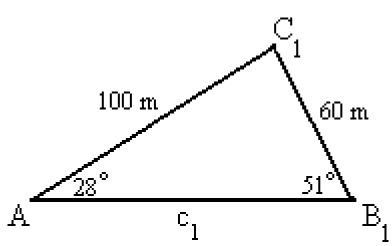
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{60}{\sin 28°} = \frac{100}{\sin B}$$
$$60 \sin B = 100 \sin 28°$$
$$\sin B = \frac{100 \sin 28°}{60}$$
$$B \approx 51°$$

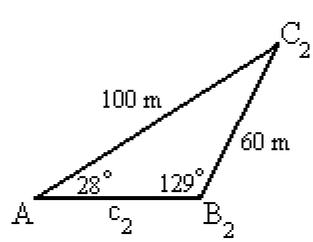
$$B_1 \approx 51°$$

The sum of B_1 and B_2 must be 180° since they would form a straight line.

$$B_2 \approx 180° - 51°$$
$$B_2 \approx 129°$$

The two triangles would be:





Example 3 (Continued):

Find the angles C_1 and C_2

Subtract the sum of other two angles from 180°

$C_1 = 180° - A - B_1$
$C_1 \approx 180° - 28° - 51°$
$C_1 \approx 101°$

$C_2 = 180° - A - B_2$
$C_2 \approx 180° - 28° - 129°$
$C_2 \approx 23°$

Find the sides c_1 and c_2

$$\frac{a}{\sin A} = \frac{c_1}{\sin C_1} \qquad \frac{a}{\sin A} = \frac{c_2}{\sin C_2}$$

$$\frac{60}{\sin 28°} \approx \frac{c_1}{\sin 101°} \qquad \frac{60}{\sin 28°} \approx \frac{c_2}{\sin 23°}$$

$$\frac{60 \sin 101°}{\sin 28°} \approx c_1 \qquad \frac{60 \sin 23°}{\sin 28°} \approx c_2$$

$c_1 \approx 125.5$ m $\qquad$ $c_2 \approx 49.9$ m

The solutions are $\angle B_1 \approx 51°$, $\angle C_1 \approx 101°$, $c_1 \approx 125.5$ m and $\angle B_2 \approx 129°$, $\angle C_2 \approx 23°$, $c_2 \approx 49.9$ m.

Example 4: Determine whether the given measurements produce one triangle, two triangles, or no triangle. If a triangle is formed then solve the triangle (or triangles) rounding the sides to the nearest tenth and the angles to the nearest degree.

$b = 67$ in, $a = 48$ in, and $\angle A = 60.5°$

Solution:

Find the height (h) of the triangle

$h = b \sin A$
$h = 67 \sin 60.5°$
$h \approx 58.3$ in

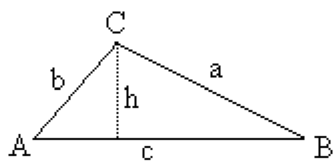
Example 4 (Continued):

Identify the case that applies (1 triangle, 2 triangles, or no triangle)

Compare sides a and b: 48 in $<$ 67 in
Compare side a and h: 48 in $<$ 58.3 in

Since side a is less than both the height and the length of side b there is no triangle formed. There is no solution for this problem since the length of side a is not long enough to form a triangle.

The Law of Sines can also be used to calculate the area of a triangle. The sine is equal to the length of the opposite side divided by the length of the hypotenuse. By drawing a line straight down for the height of the triangle, then height will become the opposite side.



We can then solve the equation formed from the sine of angle A for the height, h.

$$\sin A = \frac{h}{b}$$
$$b \times \sin A = \frac{h}{b} \times b$$
$$b \sin A = h$$

Now we can substitute this into the formula for the area of a triangle.

$$Area = \frac{1}{2} \times base \times height$$
$$Area = \frac{1}{2} \times (c) \times (b \sin A)$$
$$Area = \frac{1}{2} bc \sin A$$

Looking at the picture of the triangle you can see that the formula is one-half of the product of the sides b and c and the sine of the included angle. Using this general formula we can write the Area of the triangle in terms of the sine of any angle.

$$Area = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B = \frac{1}{2} ab \sin C$$

Example 5: Find the area of the triangle having the given measurements. Round the area to the nearest square unit.

a = 200 ft, b = 220 ft, and $\angle C = 40°$

Solution:

Substitute the known values into the appropriate area equation

$$Area = \frac{1}{2}ab\sin C$$
$$Area = \frac{1}{2}(200\,ft)(220\,ft)\sin 40°$$
$$Area = (22000\,ft^2)\sin 40°$$
$$Area \approx 14141.327\,ft^2$$

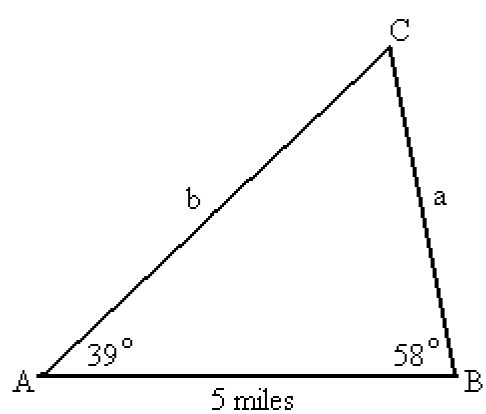
The area of the triangle is approximately 14,141 square feet.

The Law of Sines can also be used in application problems where the triangle formed is not a right triangle.

Example 6: A fire is spotted by park rangers stationed in two towers that are 5 miles apart. Using the line between them as a baseline, tower A reports the fire is at an angle of 39°, while tower B reports an angle of 58°. How far is the fire from tower B?

Solution:

Draw a diagram of the situation



Identify what needs to be found

We need to find the distance from the tower B to the fire at C which is represented by a. However, before we can find a, we need to find the measure of angle C.

Example 6 (Continued):

Find angle C

$\angle A + \angle B + \angle C = 180°$
$39° + 58° + \angle C = 180°$
$\angle C = 180° - 97°$
$\angle C = 83°$

Find a

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{a}{\sin 39°} = \frac{5}{\sin 83°}$$
$$a = \frac{5\sin 39°}{\sin 83°}$$
$$a \approx 3.17$$

The fire is approximately 3.17 miles from tower B.