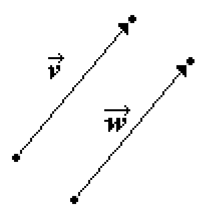
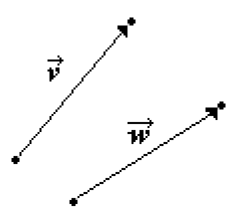
Vectors

In this section, we will be discussing vectors and scalars. Vectors are quantities that have both a magnitude and direction. A real world example of a vector can be seen in indicating the motion of a vehicle. The car is traveling northwest at 65 miles per hour. The magnitude of the vector in this example would be the car's speed of 65 miles per hour and the direction would be northwest. The use of vectors is very important in the field of Physics to represent how forces act on objects. Scalars, on the other hand, only have a magnitude but no direction. Examples of scalar quantities would be in expressing temperature or height.

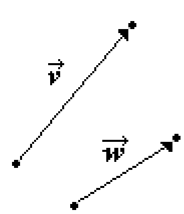
Vectors are represented by the use of rays (or directed line segments). Vectors will have an initial point (P) and a terminal point (Q). The vector is denoted with a ray over the initial and terminal points … $\overrightarrow{PQ}$ or it can be abbreviated using a letter to represent the vector … ***v***. The magnitude of a vector is denoted using a set of double vertical bars as $\left\|\overrightarrow{PQ}\right\|$ and cannot be negative since it represents the distance from the initial point to the terminal point. Two vectors are considered to be equal only if they have the same magnitude and direction.



Vectors ***v*** and ***w*** are the same since they have the same magnitude and direction.



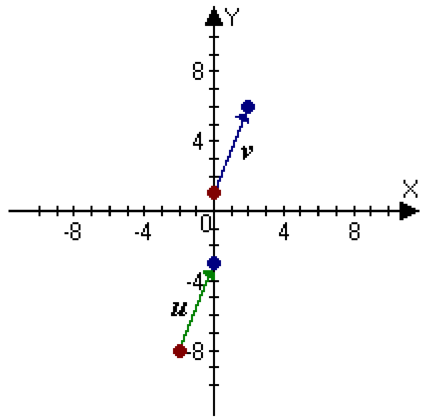
Vectors ***v*** and ***w*** are not the same because they have different directions.



Vectors ***v*** and ***w*** are not the same because they have different magnitudes and direction.

Two vectors can be compared to determine if they are the same by calculating their magnitude with the distance formula and their direction with the slope formula.

Example 1: Determine if the two vectors are equal.



Solution:

Find the magnitude of the vectors

$$\|u\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$\|u\| = \sqrt{(0-(-2))^2 + (-3-(-8))^2}$$
$$\|u\| = \sqrt{4+25}$$
$$\|u\| = \sqrt{29}$$

$$\|v\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$\|v\| = \sqrt{(2-0)^2 + (6-1)^2}$$
$$\|v\| = \sqrt{4+25}$$
$$\|v\| = \sqrt{29}$$

Find the slopes of the vectors

$$m_u = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m_u = \frac{-3-(-8)}{0-(-2)}$$
$$m_u = \frac{5}{2}$$

$$m_v = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m_v = \frac{6-1}{2-0}$$
$$m_v = \frac{5}{2}$$

The distances and slopes are the same so the two vectors would have the same magnitude and direction making them equal.

The magnitude and direction of a vector can be altered by multiplying it by a scalar quantity, *k*. The value of *k* will determine if the vector's direction changes in addition to its magnitude.

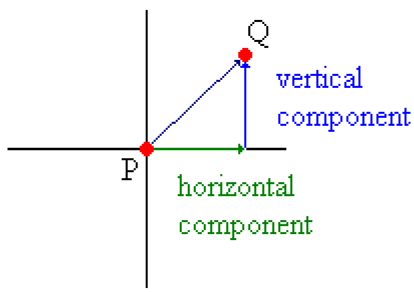
Scalar multiplication

If *k* is a real number and ***v*** is a vector, then *k* ***v*** is a scalar multiple of the vector ***v***.

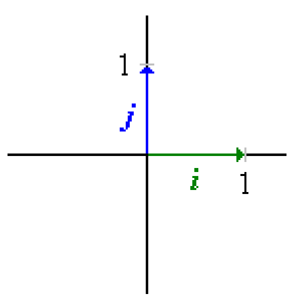
- If $k > 0$ then the resulting vector has the same direction but different magnitude
- If $k < 0$ then the resulting vector has the opposite direction and different magnitude

The magnitude of *k*v = $|k| \times \|v\|$

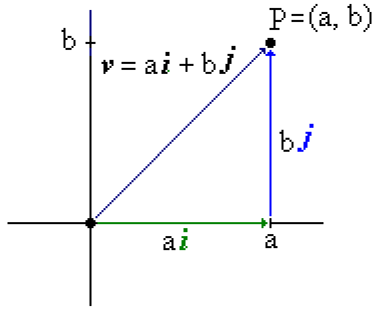
Since vectors can move in any direction they will have both a vertical and horizontal component, which can be seen if we plot a vector on the rectangular coordinate system.



In order to write a vector into its horizontal and vertical components we need a vector to represent these two directions. This is where the concept of the unit vectors ***i*** and ***j*** come into play. The unit vector ***i*** has a magnitude of 1 and its direction is along the positive x-axis of the rectangular coordinate system. The unit vector ***j*** has a magnitude of 1 and its direction is along the positive y-axis of the rectangular coordinate system.



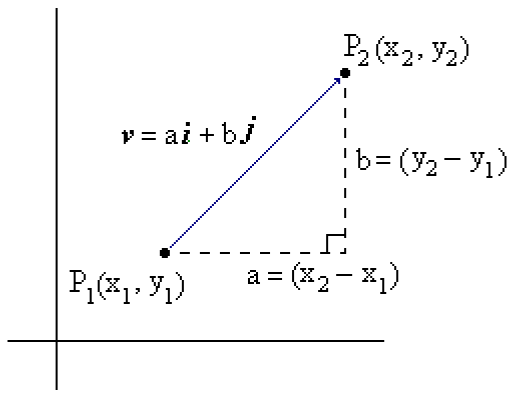
Scalar multiplication is then used to separate a vector into a sum of the two unit vectors.



For a vector from (0, 0) to (a, b):

- a is the horizontal scalar component of the vector
- b is the vertical scalar component of the vector
- ai + bj is its representation in terms of the component vectors
- $\sqrt{a^2 + b^2}$ is the vector's magnitude

Not all of the vectors will have its initial point starting at the origin. Therefore, a more general representation is needed. Using a similar vector as the one above but shifted away from the origin we can see how the vector can still be written in terms of the two units vectors ***i*** and ***j***.



$\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$
$\boldsymbol{v} = (x_2 - x_1)\,\boldsymbol{i} + (y_2 - y_1)\,\boldsymbol{j}$

$$\|v\| = \sqrt{a^2 + b^2}$$
$$\|v\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 2: Write the vector $\boldsymbol{v}$, with its initial point at the origin (0, 0) and terminal point at (-2, 3), in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$. Also, find its magnitude.

Solution:

Determine the values of a and b

Since the initial point of the vector is at the origin the coordinates of the terminal point would represent the values of a and b.

(-2, 3) = (a, b)

a = -2 and b = 3

Write the vector in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$

$\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$
$\boldsymbol{v} = -2\boldsymbol{i} + 3\boldsymbol{j}$

Find the magnitude of the vector

$$\|v\| = \sqrt{a^2 + b^2}$$
$$\|v\| = \sqrt{(-2)^2 + (3)^2}$$
$$\|v\| = \sqrt{4+9}$$
$$\|v\| = \sqrt{13}$$

Example 3: Write the vector $\boldsymbol{v}$, with its initial point at (-3, 4) and terminal point at (6, -1), in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$. Also, find its magnitude.

Solution:

Let the initial point be P_1 and the terminal point P_2

$P_1 = (x_1, y_1) = (-3, 4)$
$P_2 = (x_2, y_2) = (6, -1)$

Substitute the coordinates into the formula for the vector in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$.

$\boldsymbol{v} = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j}$
$\boldsymbol{v} = (6 - (-3))\boldsymbol{i} + (-1 - 4)\boldsymbol{j}$
$\boldsymbol{v} = 9\boldsymbol{i} - 5\boldsymbol{j}$

Example 3 (Continued):

Find the magnitude of the vector

$$\|v\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$\|v\| = \sqrt{(9)^2 + (-5)^2}$$
$$\|v\| = \sqrt{81 + 25}$$
$$\|v\| = \sqrt{106}$$

Writing vectors in terms of the unit vectors ***i*** and ***j***. makes it easy to perform operations on the vectors. The operations that will we look at are vector addition, vector subtraction, and scalar multiplication.

Vector Addition

If ***v*** = a_1 ***i*** + b_1***j*** and ***w*** = a_2 ***i*** + b_2 ***j*** then the sum of the two vectors would be found by adding the corresponding scalar components of the two vectors.

$$\boldsymbol{v} + \boldsymbol{w} = (a_1 + a_2)\,\boldsymbol{i} + (b_1 + b_2)\,\boldsymbol{j}$$

Vector Subtraction

If ***v*** = a_1 ***i*** + b_1***j*** and ***w*** = a_2 ***i*** + b_2 ***j*** then the difference of the two vectors would be found by subtracting the corresponding scalar components of the two vectors.

$$\boldsymbol{v} - \boldsymbol{w} = (a_1 - a_2)\,\boldsymbol{i} + (b_1 - b_2)\,\boldsymbol{j}$$

Scalar Multiplication

If ***v*** = a ***i*** + b***j*** and k is a real number, then the product of the vector and k would be found by distributing k to each of the scalar components of the vector.

$$k\,\boldsymbol{v} = (ka)\,\boldsymbol{i} + (kb)\,\boldsymbol{j}$$

Vector Addition Properties

Where ***u***, ***v***, and ***w*** are vectors and c and d are scalars:

$\mathbf{0} = 0\boldsymbol{i} + 0\boldsymbol{j}$	zero vector
$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$	commutative property
$(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$	associative property
$\boldsymbol{u} + 0 = 0 + \boldsymbol{u} = \boldsymbol{u}$	additive identity
$\boldsymbol{u} + (-\boldsymbol{u}) = (-\boldsymbol{u}) + \boldsymbol{u} = \mathbf{0}$	additive inverse property

Scalar Multiplication Properties

Where $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are vectors and c and d are scalars:

$(cd)\boldsymbol{u} = c(d\boldsymbol{u})$	associative property
$c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$	distributive property
$(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$	distributive property
$1\boldsymbol{u} = \boldsymbol{u}$	multiplicative identity
$0\boldsymbol{u} = \mathbf{0}$	multiplicative property of zero
$\| c\boldsymbol{u} \| = \lvert c\rvert \, \|\boldsymbol{u}\|$	magnitude property

Example 4: Find $5\boldsymbol{v} – 2\boldsymbol{w}$ if $\boldsymbol{v} = -2\boldsymbol{i} + 5\boldsymbol{j}$ and $\boldsymbol{w} = 3\boldsymbol{i} – 7\boldsymbol{j}$.

Solution:

Find $5\boldsymbol{v}$

$5\boldsymbol{v} = 5(-2\boldsymbol{i} + 5\boldsymbol{j})$
$5\boldsymbol{v} = 5(-2\boldsymbol{i}) + 5(5\boldsymbol{j})$
$5\boldsymbol{v} = -10\boldsymbol{i} + 25\boldsymbol{j}$

Find $– 2\boldsymbol{w}$

$– 2\boldsymbol{w} = – 2(3\boldsymbol{i} – 7\boldsymbol{j})$
$– 2\boldsymbol{w} = – 2(3\boldsymbol{i}) – 2(–7\boldsymbol{j})$
$– 2\boldsymbol{w} = – 6\boldsymbol{i} + 14\boldsymbol{j}$

Find $5\boldsymbol{v} – 2\boldsymbol{w}$

$5\boldsymbol{v} – 2\boldsymbol{w} = (5\boldsymbol{v}) + (–2\boldsymbol{w})$
$5\boldsymbol{v} – 2\boldsymbol{w} = (-10\boldsymbol{i} + 25\boldsymbol{j}) + (– 6\boldsymbol{i} + 14\boldsymbol{j})$
$5\boldsymbol{v} – 2\boldsymbol{w} = (-10 – 6)\boldsymbol{i} + (25 + 14)\boldsymbol{j}$
$5\boldsymbol{v} – 2\boldsymbol{w} = –16\boldsymbol{i} + 39\boldsymbol{j}$

So far the only unit vectors that we have talked about are $\boldsymbol{i}$ and $\boldsymbol{j}$. However, it is possible to have a unit vector that is in the same direction as a nonzero vector, $\boldsymbol{v}$. We can obtain the unit vector in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$ by dividing the vector $\boldsymbol{v}$ by its magnitude $\|\boldsymbol{v}\|$.

If v is a nonzero vector then the vector

$$\frac{v}{\|v\|} = \frac{ai + bj}{\|v\|} = \frac{a}{\|v\|}i + \frac{b}{\|v\|}j$$

is the unit vector that has the same direction as v.

Example 5: Find the unit vector that has the same direction as ***w*** = 4***i*** – 5***j***.

Solution:

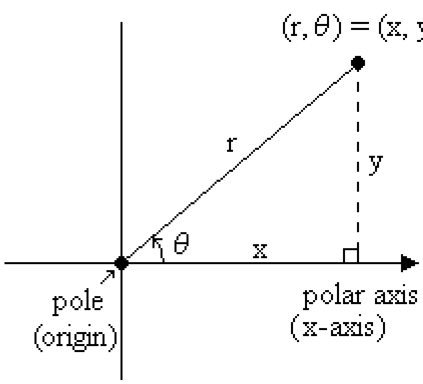
Find the magnitude

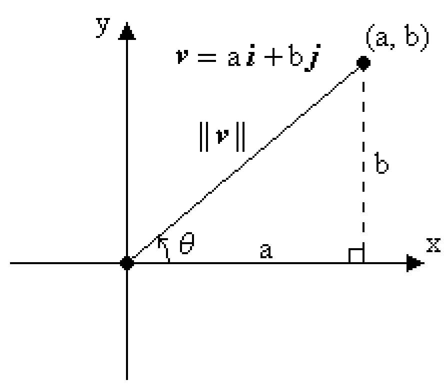
$$\|w\| = \sqrt{a^2 + b^2}$$
$$\|w\| = \sqrt{(4)^2 + (-5)^2}$$
$$\|w\| = \sqrt{16 + 25}$$
$$\|w\| = \sqrt{41}$$

Divide the vector by its magnitude

$$\frac{w}{\|w\|} = \frac{a}{\|w\|} i + \frac{b}{\|w\|} j$$
$$\frac{w}{\|w\|} = \frac{4}{\sqrt{41}} i - \frac{5}{\sqrt{41}} j$$

The same relationships that we used to convert rectangular coordinates into polar coordinates can be used to express vectors in terms of its magnitude and direction. The figures below show how this is accomplished.





$$\sin\theta = \frac{y}{r} \qquad \sin\theta = \frac{b}{\|v\|}$$

$$r \sin\theta = y \qquad \|v\| \sin\theta = b$$

$$\cos\theta = \frac{x}{r} \qquad \cos\theta = \frac{a}{\|v\|}$$

$$r \cos\theta = x \qquad \|v\| \cos\theta = a$$

We can now replace a and b with their equivalent form in terms of the magnitude and direction.

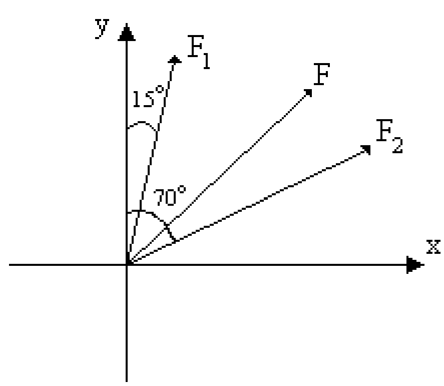
$\boldsymbol{v} = \text{a}\,\boldsymbol{i} + \text{b}\boldsymbol{j}$
$\boldsymbol{v} = \|\,\boldsymbol{v}\,\| \cos\theta\,\boldsymbol{i} + \|\,\boldsymbol{v}\,\| \sin\theta\,\boldsymbol{j}$

An example of the application of expressing the vector in terms of its magnitude and direction can be seen in the field of Physics. In Physics, forces are represented by vectors. Expressing these forces in terms of its horizontal and vertical components makes it easier to determine the net effect (resultant force) of multiple forces on an object.

Example 6: The magnitude and direction of two forces acting on an object are 110 pounds, N15°E and 180 pounds, N70°E. Find the magnitude (to the nearest pound) and the direction angle (to the nearest tenth of a degree) of the resultant force F.

Solution:

Sketch diagram of problem



Find the direction angles (from the positive x-axis) for the forces F_1 and F_2

To find the direction angle for the forces we would subtract their bearing from 90°

F_1: 90° - 15° = 75°
F_2: 90° - 70° = 20°

Express the forces in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$

$\mathbf{F_1} = \|\,\mathbf{F_1}\,\| \cos\theta\,\boldsymbol{i} + \|\,\mathbf{F_1}\,\| \sin\theta\,\boldsymbol{j}$
$\mathbf{F_1} = 110 \cos 75°\,\boldsymbol{i} + 110 \sin 75°\,\boldsymbol{j}$
$\mathbf{F_1} \approx 28.47\,\boldsymbol{i} + 106.25\,\boldsymbol{j}$

$\mathbf{F_2} = \|\,\mathbf{F_2}\,\| \cos\theta\,\boldsymbol{i} + \|\,\mathbf{F_2}\,\| \sin\theta\,\boldsymbol{j}$
$\mathbf{F_2} = 180 \cos 20°\,\boldsymbol{i} + 180 \sin 20°\,\boldsymbol{j}$
$\mathbf{F_2} \approx 169.14\,\boldsymbol{i} + 61.56\,\boldsymbol{j}$

Example 6 (Continued):

Find the resultant force

$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2}$
$\mathbf{F} \approx (28.47\,\boldsymbol{i} + 106.25\,\boldsymbol{j}) + (169.14\,\boldsymbol{i} + 61.56\,\boldsymbol{j})$
$\mathbf{F} \approx (28.47 + 169.14)\,\boldsymbol{i} + (106.25 + 61.56)\,\boldsymbol{j}$
$\mathbf{F} \approx 197.61\,\boldsymbol{i} + 167.81\,\boldsymbol{j}$

Find the magnitude of the resultant force

$$\|F_1\| = \sqrt{a^2 + b^2}$$
$$\|F_1\| \approx \sqrt{(197.61)^2 + (167.81)^2}$$
$$\|F_1\| \approx \sqrt{39049.7121 + 28160.1961}$$
$$\|F_1\| \approx \sqrt{67209.9082}$$
$$\|F_1\| \approx 259$$

Find the direction angle of the resultant force

$$\tan\theta = \frac{b}{a}$$
$$\tan\theta \approx \frac{167.81}{197.61}$$
$$\tan\theta \approx 0.8492$$
$$\theta \approx 40.3°$$