DOUBLE-ANGLE, POWER-REDUCING, AND HALF-ANGLE FORMULAS

Introduction

- Another collection of identities called double-angles and half-angles, are acquired from the sum and difference identities in section 2 of this chapter.
- By using the sum and difference identities for both sine and cosine, we are able to compile different types of double-angles and half angles
- First we are going to concentrate on the double angles and examples.

Double-Angles Identities

- Sum identity for sine:

  $\sin(x + y) = (\sin x)(\cos y) + (\cos x)(\sin y)$
  $\sin(x + x) = (\sin x)(\cos x) + (\cos x)(\sin x)$ (replace y with x)
  $\sin 2x = 2 \sin x \cos x$

  Double-angle identity for sine.

- There are three types of double-angle identity for cosine, and we use sum identity for cosine, first:

  $\cos(x + y) = (\cos x)(\cos y) - (\sin x)(\sin y)$
  $\cos(x + x) = (\cos x)(\cos x) - (\sin x)(\sin x)$ (replace y with x)
  $\cos 2x = \cos^2 x - \sin^2 x$

  First double-angle identity for cosine

- use Pythagorean identity to substitute into the first double-angle.

  $\sin^2 x + \cos^2 x = 1$
  $\cos^2 x = 1 - \sin^2 x$

  $\cos 2x = \cos^2 x - \sin^2 x$
  $\cos 2x = (1 - \sin^2 x) - \sin^2 x$ (substitute)
  $\cos 2x = 1 - 2\sin^2 x$

  Second double-angle identity for cosine.

Double-Angles Identities (Continued)

- take the Pythagorean equation in this form, $\sin^2 x = 1 - \cos^2 x$ and substitute into the First double-angle identity

  $\cos 2x = \cos^2 x - \sin^2 x$
  $\cos 2x = \cos^2 x - (1 - \cos^2 x)$
  $\cos 2x = \cos^2 x - 1 + \cos^2 x$
  $\cos 2x = 2\cos^2 x - 1$

  Third double-angle identity for cosine.

Summary of Double-Angles

- Sine:

  $\sin 2x = 2 \sin x \cos x$

- Cosine:

  $\cos 2x = \cos^2 x - \sin^2 x$
  $= 1 - 2 \sin^2 x$
  $= 2 \cos^2 x - 1$

- Tangent:

  $\tan 2x = 2 \tan x/1 - \tan^2 x$
  $= 2 \cot x/ \cot^2 x - 1$
  $= 2/\cot x - \tan x$

  tangent double-angle identity can be accomplished by applying the same methods, instead use the sum identity for tangent, first.

- Note: $\sin 2x \neq 2 \sin x$; $\cos 2x \neq 2 \cos x$; $\tan 2x \neq 2 \tan x$

Example 1: Verify, $(\sin x + \cos x)^2 = 1 + \sin 2x$:

Answer

$(\sin x + \cos x)^2 = 1 + \sin 2x$
$(\sin x + \cos x)(\sin x + \cos x) = 1 + \sin 2x$
$\sin^2 x + \sin x \cos x + \sin x \cos x + \cos^2 x = 1 + \sin 2x$
$\sin^2 x + 2\sin x \cos x + \cos^2 x = 1 + \sin 2x$ (combine like terms)
$\sin^2 x + \mathit{\sin 2x} + \cos^2 x = 1 + \sin 2x$ (substitution: double-angle identity)
$\sin^2 x + \cos^2 x + \sin 2x = 1 + \sin 2x$
$1 + \sin 2x = 1 + \sin 2x$ (Pythagorean identity)

Therefore, $1+ \sin 2x = 1 + \sin 2x$, is verifiable.

Half-Angle Identities

The alternative form of double-angle identities are the **half-angle identities**.

Sine

- To achieve the identity for sine, we start by using a double-angle identity for cosine

$\cos 2x = 1 – 2 \sin^2 x$
$\cos 2m = 1 – 2 \sin^2 m$ [replace x with m]
$\cos 2x/2 = 1 – 2 \sin^2 x/2$ [replace m with x/2]
$\cos x = 1 – 2 \sin^2 x/2$
$\sin^2 x/2 = (1 – \cos x)/2$ [solve for sin(x/2)]
$\sqrt{\sin^2 x/2} = \sqrt{[(1 – \cos x)/2]}$
$\mathbf{\sin x/2 = \pm \sqrt{[(1 – \cos x)/2]}}$

Half-angle identity for sine

- Choose the negative or positive sign according to where the x/2 lies within the Unit Circle quadrants.

Half-Angle Identities (Continued)

Cosine

- To get the half-angle identity for cosine, we begin with another double-angle identity for cosine

$\cos 2x = 2\cos^2 x - 1$
$\cos 2m = 2 \cos^2 m - 1$ [replace x with m]
$\cos 2x/2 = 2 \cos^2 x/2 - 1$ [replace m with x/2]
$\cos x = 2 \cos^2 x/2 - 1$
$\cos^2 x/2 = (1 + \cos x)/ 2$ [solve for cos (x/2)]
$\sqrt{\cos^2} \; x/2 = \sqrt{[(1 + \cos x)/ 2]}$
$\cos x/2 = \pm\sqrt{[(1 + \cos x)/ 2]}$

Half-angle identity for cosine

- Again, depending on where the x/2 within the Unit Circle, use the positive and negative sign accordingly.

Tangent

- To obtain half-angle identity for tangent, we use the quotient identity and the half-angle formulas for both cosine and sine:

$\tan x/2 = (\sin x/2)/ (\cos x/2)$	(quotient identity)
$\tan x/2 = \pm\sqrt{[(1 - \cos x)/ 2]} / \pm\sqrt{[(1 + \cos x)/ 2]}$	(half-angle identity)
$\tan x/2 = \pm\sqrt{[(1 - \cos x)/ (1 + \cos x)]}$	(algebra)

Half-angle identity for tangent

- There are easier equations to the half-angle identity for tangent equation

$\tan x/2 = \sin x/ (1 + \cos x)$ 1st easy equation

$\tan x/2 = (1 - \cos x) /\sin x$ 2nd easy equation.

Summary of Half-Angles

- Sine
  - $\sin x/2 = \pm\sqrt{[(1 - \cos x)/ 2]}$

- Cosine
  - $\cos x/2 = \pm\sqrt{[(1 + \cos x)/ 2]}$

Summary of Half-Angles (Continued)

- Tangent
  - $\tan x/2 = \pm\sqrt{(1 - \cos x)/(1 + \cos x)}$
  - $\tan x/2 = \sin x/(1 + \cos x)$
  - $\tan x/2 = (1 - \cos x)/\sin x$

- Remember, pick the positive and negative sign according to where the x/2 lies.
- Note: $\sin x/2 \neq \frac{1}{2} \sin x$; $\cos x/2 \neq \frac{1}{2} \cos x$; $\tan x/2 \neq \frac{1}{2} \tan x$

Example 2: Find exact value for, tan 30 degrees, without a calculator, and use the half-angle identities (refer to the Unit Circle).

Answer

$$
\begin{aligned}
\tan 30 \text{ degrees} &= \tan 60 \text{ degrees}/2 \\
&= \sin 60/(1 + \cos 60) \\
&= (\sqrt{3}/2)/(1+1/2) \\
&= (\sqrt{3}/2)/(3/2) \\
&= (\sqrt{3}/2)\times(2/3) \\
&= \sqrt{3}/3
\end{aligned}
$$

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