DOUBLE-ANGLE, POWER-REDUCING, AND HALF-ANGLE FORMULAS

Introduction

- Another collection of identities called double-angles and half-angles, are acquired from the sum and difference identities in section 2 of this chapter.
- By using the sum and difference identities for both sine and cosine, we are able to compile different types of double-angles and half angles
- First we are going to concentrate on the double angles and examples.

Double-Angles Identities

- Sum identity for sine:

\[
\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y) \\
\sin(2x) = 2\sin x \cos x
\]

Double-angle identity for sine.

- There are three types of double-angle identity for cosine, and we use sum identity for cosine, first:

\[
\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y) \\
\cos(2x) = \cos^2 x - \sin^2 x
\]

First double-angle identity for cosine

- use Pythagorean identity to substitute into the first double-angle.

\[
\sin^2 x + \cos^2 x = 1 \\
\cos^2 x = 1 - \sin^2 x \\
\cos 2x = \cos^2 x - \sin^2 x \\
\cos 2x = (1 - \sin^2 x) - \sin^2 x \quad \text{(substitute)} \\
\cos 2x = 1 - 2\sin^2 x
\]

Second double-angle identity for cosine.
Double-Angles Identities (Continued)

- take the Pythagorean equation in this form, \( \sin^2 x = 1 - \cos^2 x \) and substitute into the First double-angle identity

\[
\cos 2x = \cos^2 x - \sin^2 x \\
\cos 2x = \cos^2 x - (1 - \cos^2 x) \\
\cos 2x = \cos^2 x - 1 + \cos^2 x \\
\cos 2x = 2\cos^2 x - 1
\]

**Third double-angle identity for cosine.**

Summary of Double-Angles

- Sine:

\[
\sin 2x = 2 \sin x \cos x
\]

- Cosine:

\[
\cos 2x = \cos^2 x - \sin^2 x \\
= 1 - 2 \sin^2 x \\
= 2 \cos^2 x - 1
\]

- Tangent:

\[
\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \\
= \frac{2 \cot x}{\cot^2 x - 1} \\
= \frac{2}{\cot x - \tan x}
\]

tangent double-angle identity can be accomplished by applying the same methods, instead use the sum identity for tangent, first.

- Note: \( \sin 2x \neq 2 \sin x; \cos 2x \neq 2 \cos x; \tan 2x \neq 2 \tan x \)
**Example 1:** Verify, \((\sin x + \cos x)^2 = 1 + \sin 2x:\)

Answer

\[(\sin x + \cos x)^2 = 1 + \sin 2x\]

\[(\sin x + \cos x)(\sin x + \cos x) = 1 + \sin 2x\]

\[\sin^2 x + \sin x \cos x + \sin x \cos x + \cos^2 x = 1 + \sin 2x\]  
(\text{combine like terms})

\[\sin^2 x + \sin 2x + \cos^2 x = 1 + \sin 2x\]  
(substitution: double-angle identity)

\[\sin^2 x + \cos^2 x + \sin 2x = 1 + \sin 2x\]  
(Pythagorean identity)

Therefore, \(1 + \sin 2x = 1 + \sin 2x\), is verifiable.

**Half-Angle Identities**

The alternative form of double-angle identities are the **half-angle identities**.

**Sine**

- To achieve the identity for sine, we start by using a double-angle identity for cosine

\[
\cos 2x = 1 - 2 \sin^2 x \\
\cos 2m = 1 - 2 \sin^2 m \quad \text{[replace } x \text{ with } m]\]

\[
\cos 2x/2 = 1 - 2 \sin^2 x/2 \quad \text{[replace } m \text{ with } x/2]\]

\[
\cos x = 1 - 2 \sin^2 x/2 \\
\sin^2 x/2 = (1 - \cos x)/2 \quad \text{[solve for } \sin(x/2)\text{]}\]

\[
\quad \sqrt{\sin^2 x/2} = \pm \sqrt{(1 - \cos x)/2}\]

**Half-angle identity for sine**

- Choose the negative or positive sign according to where the \(x/2\) lies within the Unit Circle quadrants.

by Shavana Gonzalez
Half-Angle Identities (Continued)

Cosine

- To get the half-angle identity for cosine, we begin with another double-angle identity for cosine

\[ \cos 2x = 2 \cos^2 x - 1 \]
\[ \cos 2m = 2 \cos^2 m - 1 \text{ [replace x with m]} \]
\[ \cos 2x/2 = 2 \cos^2 x/2 - 1 \text{ [replace m with x/2]} \]
\[ \cos x = 2 \cos^2 x/2 - 1 \]
\[ \cos^2 x/2 = (1 + \cos x)/ 2 \text{ [solve for } \cos (x/2)\text{]} \]
\[ \sqrt{\cos^2 x/2} = \pm \sqrt{(1 + \cos x)/ 2} \]

**Half-angle identity for cosine**

- Again, depending on where the x/2 within the Unit Circle, use the positive and negative sign accordingly.

Tangent

- To obtain half-angle identity for tangent, we use the quotient identity and the half-angle formulas for both cosine and sine:

\[ \tan x/2 = \frac{\sin x/2}{\cos x/2} \] (quotient identity)
\[ \tan x/2 = \pm \sqrt{[(1 - \cos x)/ 2] / \pm \sqrt{[(1 + \cos x)/ 2]}} \] (half-angle identity)
\[ \tan x/2 = \pm \sqrt{[(1 - \cos x)/ (1 + \cos x)]} \] (algebra)

**Half-angle identity for tangent**

- There are easier equations to the half-angle identity for tangent equation

\[ \tan x/2 = \frac{\sin x}{1 + \cos x} \] 1st easy equation
\[ \tan x/2 = \frac{1 - \cos x}{\sin x} \] 2nd easy equation.

Summary of Half-Angles

- Sine
  - \[ \sin x/2 = \pm \sqrt{[(1 - \cos x)/ 2]} \]

- Cosine
  - \[ \cos x/2 = \pm \sqrt{[(1 + \cos x)/ 2]} \]
Summary of Half-Angles (Continued)

- **Tangent**
  - \(\tan \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}}\)
  - \(\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}\)
  - \(\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x}\)

- Remember, pick the positive and negative sign according to where the \(x/2\) lies.
- Note: \(\sin x/2 \neq \frac{1}{2} \sin x\); \(\cos x/2 \neq \frac{1}{2} \cos x\); \(\tan x/2 \neq \frac{1}{2} \tan x\)

**Example 2:** Find exact value for, \(\tan 30\) degrees, without a calculator, and use the half-angle identities (refer to the Unit Circle).

**Answer**

\[
\tan 30\text{ degrees} = \tan 60\text{ degrees} / 2 \\
= \frac{\sin 60}{1 + \cos 60} \\
= \frac{\sqrt{3}/2}{1+1/2} \\
= \frac{\sqrt{3}/2}{3/2} \\
= (\sqrt{3}/2) \times (2/3) \\
= \sqrt{3}/3
\]