Product-to-Sum and Sum-to-Product Identities

The product-to-sum identities are used to rewrite the product between sines and/or cosines into a sum or difference. These identities are derived by adding or subtracting the sum and difference formulas for sine and cosine that were covered in an earlier section.

For example, $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ and $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. If we were to subtract $\sin(\alpha - \beta)$ from $\sin(\alpha + \beta)$ we could derived the product-to-sum identity for the product of $\cos\alpha\cos\beta$.

$$\begin{array}{rl} \sin(\alpha + \beta) = & \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \underline{-\ \sin(\alpha - \beta) =} & \underline{-[\sin\alpha\cos\beta - \cos\alpha\sin\beta]} \end{array}$$

$$\begin{array}{rl} \sin(\alpha + \beta) = & \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \underline{-\ \sin(\alpha - \beta) =} & \underline{-\sin\alpha\cos\beta + \cos\alpha\sin\beta} \\ \sin(\alpha + \beta) - \sin(\alpha - \beta) = & \quad 0 \quad + 2\cos\alpha\sin\beta \end{array}$$

$$2\cos\alpha\sin\beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$$

Now multiply both sides of the equation by ½

$$\tfrac{1}{2}(2\cos\alpha\sin\beta) = \tfrac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

Product-to-Sum Identities

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

Example 1: Express the product of $\cos 3x \cos 5x$ as a sum or difference.

Solution:

Identify which identity will be used

The given product is a product of two cosines so the $\cos\alpha\cos\beta$ identity would be used.

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

Example 1 (Continued):

Apply the product-to-sum identity for cos α cos β

$\alpha = 3x$ and $\beta = 5x$

$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$
$\cos 3x \cos 5x = \frac{1}{2} [\cos(3x - 5x) + \cos(3x + 5x)]$
$\cos 3x \cos 5x = \frac{1}{2} [\cos(-2x) + \cos(8x)]$

Apply the even/odd identity for cos (-x)

$\cos(-x) = \cos x$

$\cos 3x \cos 5x = \frac{1}{2} [\cos 2x + \cos 8x]$

The purpose of the sum-to-product identities is the reverse of the product-to-sum identities. These identities are used to rewrite the sum or difference of sines and/or cosines in a product. If you wanted to verify the identity, we would use the product-to-sum identities.

Sum-to-Product Identities

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}$$
$$\sin \alpha - \sin \beta = 2 \sin \frac{\alpha-\beta}{2} \cos \frac{\alpha+\beta}{2}$$
$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}$$
$$\cos \alpha - \cos \beta = -2 \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}$$

Example 2: Express the difference of sin 2x – sin x as a product.

Solution:

Identify which identity will be used

The given difference is the difference between two sines so the sin α – sin β identity would be used.

$$\sin \alpha - \sin \beta = 2 \sin \frac{\alpha-\beta}{2} \cos \frac{\alpha+\beta}{2}$$

Example 2 (Continued):

Apply the sum-to-product identity for $\sin \alpha - \sin \beta$

$\alpha = 2x$ and $\beta = x$

$$\sin \alpha - \sin \beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

$$\sin 2x - \sin x = 2 \sin \frac{2x - x}{2} \cos \frac{2x + x}{2}$$

$$\sin 2x - \sin x = 2 \sin \frac{x}{2} \cos \frac{3x}{2}$$

Example 3: Verify the following identity.

$$\frac{\sin 4x + \sin 6x}{\sin 4x - \sin 6x} = -\tan 5x \cot x$$

Solution:

Apply the sum-to-product identity for $\sin \alpha + \sin \beta$ to the numerator

$\alpha = 4x$ and $\beta = 6x$

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin 4x + \sin 6x = 2 \sin \frac{4x + 6x}{2} \cos \frac{4x - 6x}{2}$$

$$\sin 4x + \sin 6x = 2 \sin 5x \cos (-x)$$

$$\cos (-x) = \cos x$$

$$\sin 4x + \sin 6x = 2 \sin 5x \cos x$$

Example 3 (Continued):

Apply the sum-to-product identity for sin α – sin β to the denominator

α = 4x and β = 6x

$$\sin \alpha - \sin \beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

$$\sin 4x - \sin 6x = 2 \sin \frac{4x - 6x}{2} \cos \frac{4x + 6x}{2}$$

$$\sin 4x - \sin 6x = 2 \sin (-x) \cos 5x$$

$$\sin (-x) = -\sin x$$

$$\sin 4x - \sin 6x = -2 \sin x \cos 5x$$

Substitute the products into the identity

$$\frac{\sin 4x + \sin 6x}{\sin 4x - \sin 6x} = -\tan 5x \cot x$$

$$\frac{2 \sin 5x \cos x}{-2 \sin x \cos 5x} = -\tan 5x \cot x$$

Rearrange the factors, separate into individual fractions, and reduce

$$\frac{2}{-2} \cdot \frac{\sin 5x}{\cos 5x} \cdot \frac{\cos x}{\sin x} = -\tan 5x \cot x$$

$$-\tan 5x \cot x = -\tan 5x \cot x$$