Product-to-Sum and Sum-to-Product Identities

The product-to-sum identities are used to rewrite the product between sines and/or cosines into a sum or difference. These identities are derived by adding or subtracting the sum and difference formulas for sine and cosine that were covered in an earlier section.

For example, \( \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \) and \( \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \). If we were to subtract \( \sin(\alpha - \beta) \) from \( \sin(\alpha + \beta) \) we could derived the product-to-sum identity for the product of \( \cos \alpha \cos \beta \).

\[
\begin{align*}
\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
- \sin(\alpha - \beta) &= -[\sin \alpha \cos \beta - \cos \alpha \sin \beta]
\end{align*}
\]

\[
\begin{align*}
\sin(\alpha + \beta) - \sin (\alpha - \beta) &= 0 + 2 \cos \alpha \sin \beta \\
2 \cos \alpha \sin \beta &= \sin(\alpha + \beta) - \sin (\alpha - \beta)
\end{align*}
\]

Now multiply both sides of the equation by \( \frac{1}{2} \)

\[
\frac{1}{2} (2 \cos \alpha \sin \beta) = \frac{1}{2} [\sin(\alpha + \beta) - \sin (\alpha - \beta)]
\]

\[
\cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin (\alpha - \beta)]
\]

<table>
<thead>
<tr>
<th>Product-to-Sum Identities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)] )</td>
</tr>
<tr>
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</tr>
</tbody>
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**Example 1:** Express the product of \( \cos 3x \cos 5x \) as a sum or difference.

**Solution:**

Identify which identity will be used

The given product is a product of two cosines so the \( \cos \alpha \cos \beta \) identity would be used.

\( \cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)] \)
Example 1 (Continued):

Apply the product-to-sum identity for \( \cos \alpha \cos \beta \)

\[ \alpha = 3x \text{ and } \beta = 5x \]

\[ \cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)] \]
\[ \cos 3x \cos 5x = \frac{1}{2} [\cos(3x - 5x) + \cos(3x + 5x)] \]
\[ \cos 3x \cos 5x = \frac{1}{2} [\cos(-2x) + \cos(8x)] \]

Apply the even/odd identity for \( \cos(-x) \)

\[ \cos(-x) = \cos x \]

\[ \cos 3x \cos 5x = \frac{1}{2} [\cos 2x + \cos 8x] \]

The purpose of the sum-to-product identities is the reverse of the product-to-sum identities. These identities are used to rewrite the sum or difference of sines and/or cosines in a product. If you wanted to verify the identity, we would use the product-to-sum identities.

### Sum-to-Product Identities

\[
\begin{align*}
\sin \alpha + \sin \beta &= 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \\
\sin \alpha - \sin \beta &= 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2} \\
\cos \alpha + \cos \beta &= 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \\
\cos \alpha - \cos \beta &= -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} 
\end{align*}
\]

Example 2: Express the difference of \( \sin 2x - \sin x \) as a product.

Solution:

Identify which identity will be used

The given difference is the difference between two sines so the \( \sin \alpha - \sin \beta \) identity would be used.

\[ \sin \alpha - \sin \beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2} \]
Example 2 (Continued):

Apply the sum-to-product identity for \( \sin \alpha - \sin \beta \)

\[ \alpha = 2x \text{ and } \beta = x \]

\[ \sin \alpha - \sin \beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2} \]

\[ \sin 2x - \sin x = 2 \sin \frac{2x - x}{2} \cos \frac{2x + x}{2} \]

\[ \sin 2x - \sin x = 2 \sin \frac{x}{2} \cos \frac{3x}{2} \]

Example 3: Verify the following identity.

\[ \frac{\sin 4x + \sin 6x}{\sin 4x - \sin 6x} = -\tan 5x \cot x \]

Solution:

Apply the sum-to-product identity for \( \sin \alpha + \sin \beta \) to the numerator

\[ \alpha = 4x \text{ and } \beta = 6x \]

\[ \sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \]

\[ \sin 4x + \sin 6x = 2 \sin \frac{4x + 6x}{2} \cos \frac{4x - 6x}{2} \]

\[ \sin 4x + \sin 6x = 2 \sin 5x \cos (-x) \]

\[ \cos (-x) = \cos x \]

\[ \sin 4x + \sin 6x = 2 \sin 5x \cos x \]
Example 3 (Continued):

Apply the sum-to-product identity for $\sin \alpha - \sin \beta$ to the denominator

$\alpha = 4x$ and $\beta = 6x$

$\sin \alpha - \sin \beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$

$\sin 4x - \sin 6x = 2 \sin \frac{4x - 6x}{2} \cos \frac{4x + 6x}{2}$

$\sin 4x - \sin 6x = 2 \sin (-x) \cos 5x$

$\sin (-x) = -\sin x$

$\sin 4x - \sin 6x = -2 \sin x \cos 5x$

Substitute the products into the identity

$\frac{\sin 4x + \sin 6x}{\sin 4x - \sin 6x} = -\tan 5x \cot x$

$\frac{2 \sin 5x \cos x}{-2 \sin x \cos 5x} = -\tan 5x \cot x$

Rearrange the factors, separate into individual fractions, and reduce

$\frac{2 \sin 5x \cos x}{-2 \cos 5x \sin x} = -\tan 5x \cot x$

$-\tan 5x \cot x = -\tan 5x \cot x$