Review Exercise Set 22

Exercise 1: Find the vertices and foci for the given hyperbola. Sketch the graph.

\[ \frac{y^2}{9} - \frac{x^2}{25} = 1 \]

Exercise 2: Find the equation of a hyperbola in standard form with foci at (0, -5) and (0, 5) and vertices at (0, -2) and (0, 2).

Exercise 3: Find the foci and equations of the asymptotes.

\[ \frac{x^2}{16} - \frac{y^2}{36} = 1 \]
Exercise 4: Convert the given equation into the standard form of a hyperbola by completing the square of $x$ and $y$. Find the foci and equations of the asymptotes. Sketch the graph.

$$9y^2 - x^2 + 54y + 4x + 68 = 0$$

Exercise 5: The perpendicular cross section of a nuclear power station cooling tower form two branches of a hyperbola. Suppose the central cross section of the tower is modeled by the hyperbola $2500x^2 - 625(y - 60)^2 = 1,000,000$ ($x$ and $y$ are in feet). What is the minimum distance between the sides of the tower?
Review Exercise Set 22 Answer Key

Exercise 1: Find the vertices and foci for the given hyperbola.

\[
\frac{y^2}{9} - \frac{x^2}{25} = 1
\]

Identify the transverse axis

Since the $y^2$ term is preceded by a plus sign, the transverse axis will be along the $y$-axis and the hyperbola would follow the equation of:

\[
\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1
\]

Find the vertices

\[a^2 = 9\]
\[a = 3\]

The vertices (0, -a) and (0, a) are (0, -3) and (0, 3)

Find the foci

\[c^2 = a^2 + b^2\]
\[c^2 = 9 + 25\]
\[c^2 = 34\]
\[c = \sqrt{34}\]

The foci (0, -c) and (0, c) are (0, $-\sqrt{34}$) and (0, $\sqrt{34}$)

Exercise 2: Find the equation of a hyperbola in standard form with foci at (0, -5) and (0, 5) and vertices at (0, -2) and (0, 2).

Identify the transverse axis

Since the foci are located on the $y$-axis, the transverse axis will be along the $y$-axis and the hyperbola would follow the equation of:

\[
\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1
\]
Exercise 2 (Continued):

Find the value of $a^2$

The vertices are at (0, -2) and (0, 2) so the value of $a = 2$

$$a^2 = 2^2$$
$$a^2 = 4$$

Find the value of $b^2$

The foci are at (0, -5) and (0, 5) so the value of $c = 5$

$$c^2 = a^2 + b^2$$
$$5^2 = 4 + b^2$$
$$25 - 4 = b^2$$
$$b^2 = 21$$

Substitute the values of $a^2$ and $b^2$ into the equation of a hyperbola

$$\frac{y^2}{4} - \frac{x^2}{21} = 1$$

Exercise 3: Find the foci and equations of the asymptotes.

$$\frac{x^2}{16} - \frac{y^2}{36} = 1$$

Identify the transverse axis

Since the $x^2$ term is preceded by a plus sign, the transverse axis will be along the x-axis and the hyperbola would follow the equation of:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Find the foci

Since the transverse axis is along the x-axis the foci would be in the form of (-c, 0) and (c, 0)

$$c^2 = a^2 + b^2$$
$$c^2 = 16 + 36$$
$$c^2 = 52$$
$$c = \sqrt{52}$$
$$c = 2\sqrt{13}$$

The foci (-c, 0) and (c, 0) are (-$2\sqrt{13}$, 0) and ($2\sqrt{13}$, 0)
Exercise 3 (Continued):

Find the equations of the asymptotes

\[ b^2 = 36 \]
\[ b = 6 \]

\[ y = \pm \frac{b}{a} x \]
\[ = \pm \frac{6}{4} x \]
\[ = \pm \frac{3}{2} x \]

Exercise 4: Convert the given equation into the standard form of a hyperbola by completing the square of x and y. Find the vertices, foci, and equations of the asymptotes. Sketch the graph.

\[ 9y^2 - x^2 + 54y + 4x + 68 = 0 \]

Rewrite the equation grouping the x-terms and y-terms on the left and the constant on the right

\[ (9y^2 + 54y) + (-x^2 + 4x) = -68 \]

Factor so that the coefficients of the \( x^2 \) and \( y^2 \) terms is 1

\[ 9(y^2 + 6y) - (x^2 - 4x) = -68 \]

Complete the square and simplify the equation

\[ 9 \left( y^2 + \frac{6y}{2} + \left( \frac{6}{2} \right)^2 \right) - \left( x^2 - 4x + \left( \frac{-4}{2} \right)^2 \right) = -68 + 9 \left( \frac{6}{2} \right)^2 - 1 \left( \frac{-4}{2} \right)^2 \]

\[ 9 \left( y^2 + 6y + 3^2 \right) - \left( x^2 - 4x + 2^2 \right) = -68 + 9(3)^2 - 1(-2)^2 \]

\[ 9 \left( y + 3 \right)^2 - (x - 2)^2 = -68 + 81 - 4 \]
\[ 9 \left( y + 3 \right)^2 - (x - 2)^2 = 9 \]
\[ \left( y + 3 \right)^2 - \left( \frac{x - 2}{3} \right)^2 = \frac{9}{9} \]
\[ \left( y + 3 \right)^2 - \left( \frac{x - 2}{3} \right)^2 = 1 \]
Exercise 4 (Continued):

Identify the transverse axis

Since the $y^2$ term is preceded by a plus sign, the transverse axis will be along the $y$-axis.

Find the center

$$\frac{(y-(-3))^2}{1} - \frac{(x-2)^2}{9} = 1$$

$(h, k) = (2, -3)$

Find the vertices

$a^2 = 1$

$a = 1$

The vertices $(h, k - a)$ and $(h, k + a)$ are:

$(2, -3 - 1) = (2, -4)$ and $(2, -3 + 1) = (2, -2)$

Find the foci

Since the transverse axis is along the $y$-axis the foci would be in the form of $(h, k - c)$ and $(h, k + c)$

$c^2 = a^2 + b^2$

$c^2 = 1 + 9$

$c^2 = 10$

$c = \sqrt{10}$

The foci $(h, k - c)$ and $(h, k + c)$ are:

$(2, -3 - \sqrt{10})$ and $(2, -3 + \sqrt{10})$

Find the equations of the asymptotes

$b^2 = 9$

$b = 3$

$$y = \pm \frac{a}{b} x$$

$$= \pm \frac{1}{3} x$$
Exercise 4 (Continued):

Since the center is at (2, -3), the asymptotes would be shifted 2 units to the right and 3 units down

\[ y + 3 = \pm \frac{1}{3} (x - 2) \]

or

\[ y + 3 = \frac{1}{3} (x - 2) \]
\[ y + 3 = \frac{1}{3} x - \frac{2}{3} \]
\[ y = \frac{1}{3} x - \frac{2}{3} - 3 \]
\[ y = \frac{1}{3} x - \frac{11}{3} \]

\[ y + 3 = -\frac{1}{3} (x - 2) \]
\[ y + 3 = -\frac{1}{3} x + \frac{2}{3} \]
\[ y = -\frac{1}{3} x + \frac{2}{3} - 3 \]
\[ y = -\frac{1}{3} x - \frac{7}{3} \]

Sketch the graph
Exercise 5: The perpendicular cross section of a nuclear power station cooling tower form two branches of a hyperbola. Suppose the central cross section of the tower is modeled by the hyperbola $2500x^2 - 625(y - 60)^2 = 1,000,000$ (x and y are in feet). What is the minimum distance between the sides of the tower?

Rewrite the equation into standard form

\[
\frac{2500x^2}{1,000,000} - \frac{625(y - 60)^2}{1,000,000} = 1
\]

Draw a diagram of the problem

Find the value of a

\[
a^2 = 400
\]

\[
a = 20
\]

Find the minimum distance between the sides of the tower

The minimum distance would be the distance between the two vertices which would be twice the value of a

\[
2a = 2(20) = 40
\]

The minimum distance between the sides of the tower is 40 feet.