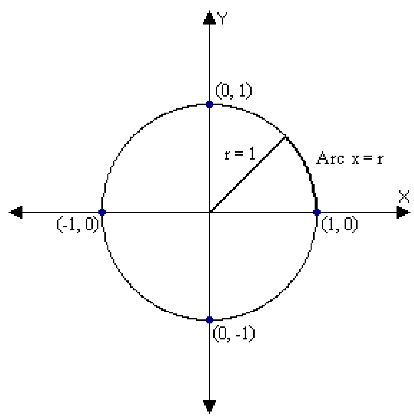
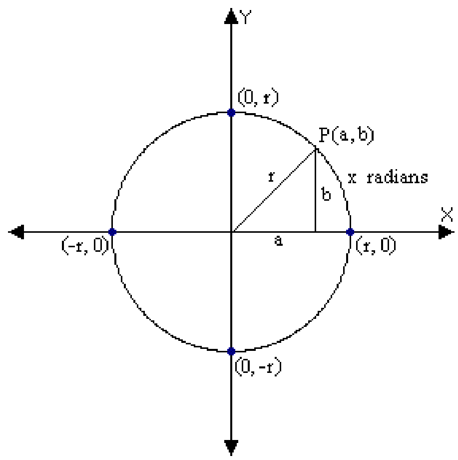
Graphs of Sine and Cosine Functions

In previous sections, we defined the trigonometric or circular functions in terms of the movement of a point around the circumference of a unit circle, or the angle formed by the rotation of a line about a point. We determined that the distance traveled by a point moving all the way around the circumference of the unit circle with radius $r = 1$ was equal to 2π, and we defined the radian as the length of the arc on the unit circle equal to the radius, as shown below. The distance traveled around the circumference of any circle is thus equal to 2π radians.



On the unit circle as well as any circle centered in a rectangular coordinate system, the trigonometric functions were defined in terms of the horizontal and vertical components of a point on the circle as well as the radius of the circle. Thus for the circle shown below with radius r, circular point P having horizontal and vertical components a and b, respectively, and x being the distance in radians traveled around the circle to point P, the trigonometric functions were defined as follows:

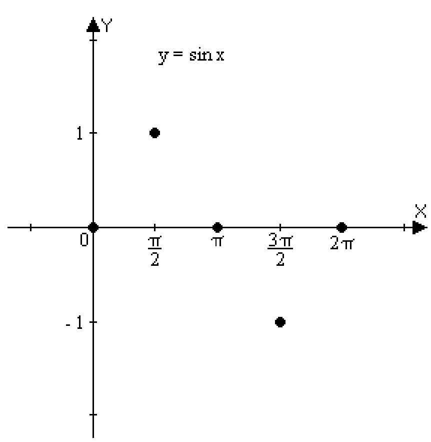


$$\sin x = \frac{b}{r} \qquad \cos x = \frac{a}{r}$$

$$\tan x = \frac{b}{a},\ \ a \neq 0 \qquad \cot x = \frac{a}{b},\ \ b \neq 0$$

$$\sec x = \frac{r}{a},\ \ a \neq 0 \qquad \csc x = \frac{r}{b},\ \ b \neq 0$$

We would now like to graph the trigonometric functions of the arc x as the circular point moves around the circumference of the circle. On the graph, the vertical axis y represents the value of the trigonometric function and the horizontal axis represents the length of the arc x in radians. We may begin by plotting the graph of the sine function, that is, $y = \sin x$, as shown below.



If we start with $x = 0$ and move in a counter-clockwise (positive) direction around the circle of Fig. 2, the value of x is plotted along the horizontal axis to the right of the y-axis in Fig. 3. If we were to move in a clockwise (negative) direction, the value of x would be plotted to the left of the y-axis. For $x = 0$, the vertical component b of the circular point P is equal to 0, so

$$y = \sin x = \sin 0 = \frac{b}{r} = \frac{0}{r} = 0$$

Our first point on the graph of $y = \sin x$ is thus $y = \sin 0 = 0$. This is the point (0, 0) located at the origin of the graph. If the circular point now moves a quarter of the way around the circle in a positive direction, arc x equals $\pi/2$ radians, the vertical component b of the circular point is equal to the radius r, and

$$\sin x = \sin \frac{\pi}{2} = \frac{b}{r} = \frac{r}{r} = 1$$

Our second point on the graph is thus $y = \sin \pi/2 = 1$, located at $(\pi/2, 1)$. Continuing to move another quarter of the way around the circle, x equals π radians, the vertical component b is again equal to 0, and

$$\sin x = \sin \pi = \frac{b}{r} = \frac{0}{r} = 0$$

Our third point on the graph is thus $y = \sin \pi = 0$, located at $(\pi, 0)$. Continuing another quarter turn, x equals $3\pi/2$ radians, and the vertical component b is again equal in length to the radius r but is now negative. (Remember from the previous sections that the radius r is always a positive number.) Thus,

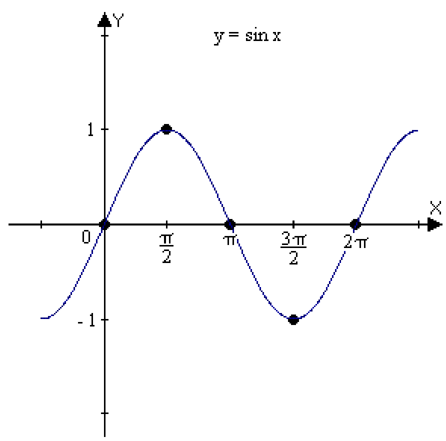
$$\sin x = \sin \frac{3\pi}{2} = \frac{b}{r} = \frac{-r}{r} = -1$$

Our fourth point on the graph is then y = $\sin 3\pi/2 = -1$, located at $(3\pi/2, -1)$. Continuing around the final quarter of the circle, x equals 2π radians, and like $x = 0$, the vertical component b is equal to 0, and

$$\sin x = \sin 2\pi = \frac{b}{r} = \frac{0}{r} = 0$$

Our fifth point on the graph is then y = sin 2π = 0.

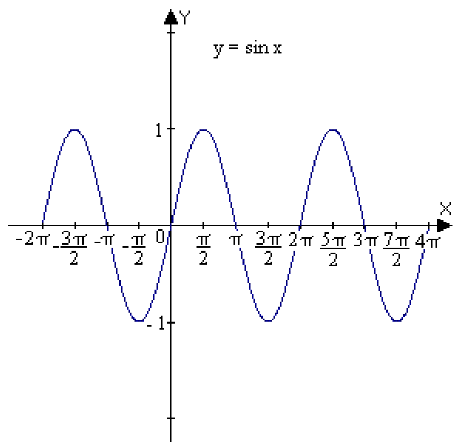
Having found these five critical points on the graph of the function $y = \sin x$, we would now like to fill in the graph between these points, as shown in Fig. 4 below. We should note that the function $y = \sin x = b/r$ has a maximum value of 1 at $x = \pi/2$, since at that point b is equal to r. The magnitude of b can never exceed r, since b is always just the vertical component of the radius r. The function has a minimum value of –1 at x = 3π/2, since at that point b is equal to $-r$.



As the circular point moves around the circle of Fig. 2, the value of the function $y = \sin x = b/r$ is dependent only on the value of b, since the value of r remains constant. For the first quarter of the circle, the function thus increases from 0 to 1 as b increases from 0 to r. For the second quarter of the circle, the function decreases from 1 to 0 as b decreases from r to 0. For the third quarter of the circle, the function decreases from 0 to –1 as b decreases from 0 to $-r$. And for the final fourth quarter of the circle, the function increases from –1 to 0 as b increases from $-r$ to 0.

Observation of the changes in the vertical component b as the circular point moves around the circle of Fig. 2 can explain the shape of the sine function's graph in Fig. 4. Near $x = 0$ and $x = \pi$, b (and therefore $y = \sin x = b/r$) is increasing or decreasing the fastest for a given change in x, since here most of the movement of the point is vertical, while near $x = \pi/2$ and $x = 3\pi/2$, b is increasing or decreasing the slowest, since here most of the movement of the point is horizontal.

Having gone completely around the circle one time, we can say we have completed one **cycle** on the graph of the function. If we were to continue around the circle a second or more times, the values of $y = \sin x$ would be exactly the same, and the second or third cycle would look just like the first cycle. This kind of function is called a **periodic function**, since the pattern continues to repeat itself. The distance along the horizontal axis representing one cycle is called the **period** of the function. In this case, the period of $y = \sin x$ is equal to 2π radians. Shown below are two positive cycles and one negative cycle of $y = \sin x$.



Now we can move on to graph the next trigonometric function, the cosine function $y = \cos x$. This function is similar in all respects to the sine function, except that now we are focusing on the horizontal component a of the circular point P as it moves around the circle of Fig. 2 instead of the vertical component b.

For $x = 0$, a is equal to r, so $y = \cos x = \cos 0 = \frac{a}{r} = \frac{r}{r} = 1$.

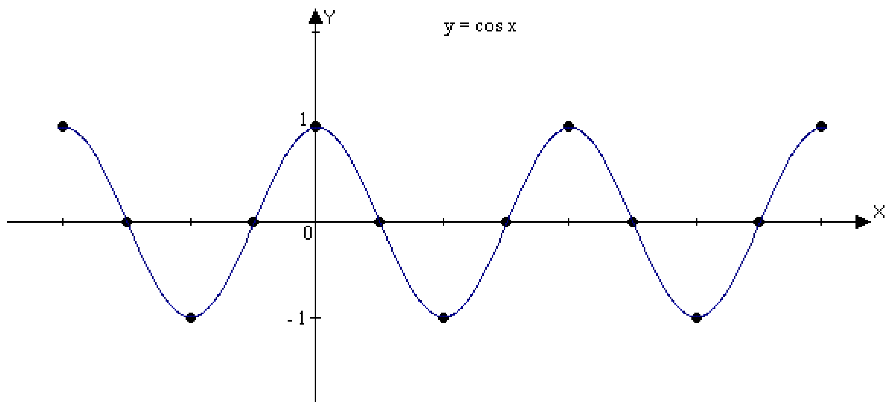
For x = $\pi/2$, a is equal to 0, and $\cos x = \cos\frac{\pi}{2} = \frac{a}{r} = \frac{0}{r} = 0$.

For x = π, a is equal to $-r$, and $\cos x = \cos\pi = \frac{a}{r} = \frac{-r}{r} = -1$.

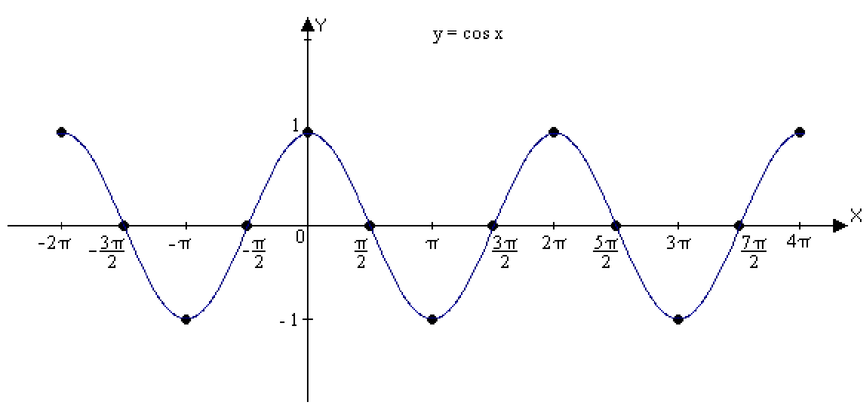
For x = $3\pi/2$, a is equal to 0, and $\cos x = \cos\frac{3\pi}{2} = \frac{a}{r} = \frac{0}{r} = 0$.

Finally, for x = 2π, a is again equal to r, and $\cos x = \cos 2\pi = \frac{a}{r} = \frac{r}{r} = 1$.

These five points on the graph of $y = \cos x$ are shown below.



Filling in the rest of the graph between these points can be accomplished in the same manner as for the sine function. The value of $y = \cos x = a/r$ as the circular point moves around the circle is simply a function of the horizontal component a of the circular point, since the radius r remains constant. The shape of the graph turns out to be exactly the same as the sine function graph, the only difference in the two being that the cosine graph starts out at $y = 1$ for $x = 0$ while the sine graph started out at $y = 0$ for $x = 0$. The period of the two functions is the same, 2π radians, since both functions begin to repeat themselves after the circular point finishes one complete revolution. Positive and negative cycles of $y = \cos x$ are shown below.

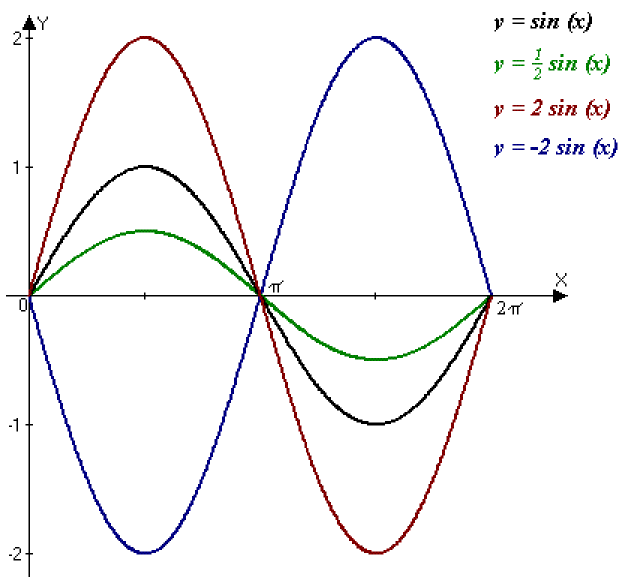


We would now like to expand the basic sine and cosine trigonometric functions to include more general cases of these functions, that is, functions which have been transformed in various ways.

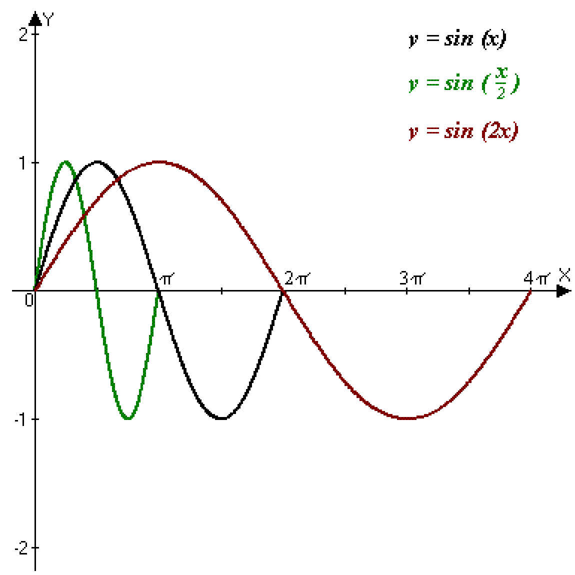
Sine and cosine functions, which are sometimes said to be **sinusoidal** in nature, represent the behavior of many phenomena in the physical world. These include, but are certainly not limited to, sound and radio waves, light rays, other forms of radiation, alternating electric current, and vibrations.

Instead of the simple basic functions $y = \sin x$ and $y = \cos x$, we would like to examine a more general form of the functions, for example $y = k + A \sin (Bx + C)$. This is still a sine function of x, but the constants allow for a number of transformations to the basic function.

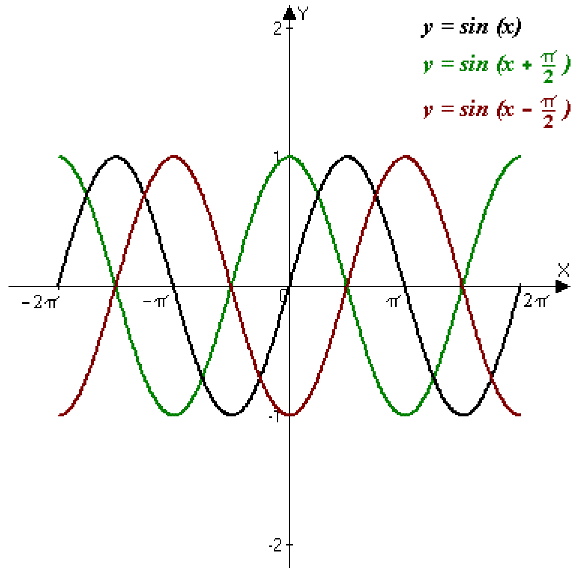
Remember from the previous section that the maximum value of $y = \sin x$ is 1, and this occurs at $x = \pi / 2$ radians. If we multiply the function by the constant A, where A is any positive number, the maximum value of the function will be A at $x = \pi / 2$. If A is a negative number, then the function will be reflected about the x-axis. The graph of $y = A \sin x$ can be drawn by multiplying each value of the original function $y = \sin x$ by A. The absolute value of the constant A is called the **amplitude** of the function. Some graphical examples of the function $y = A \sin x$ for different values of A are shown below.



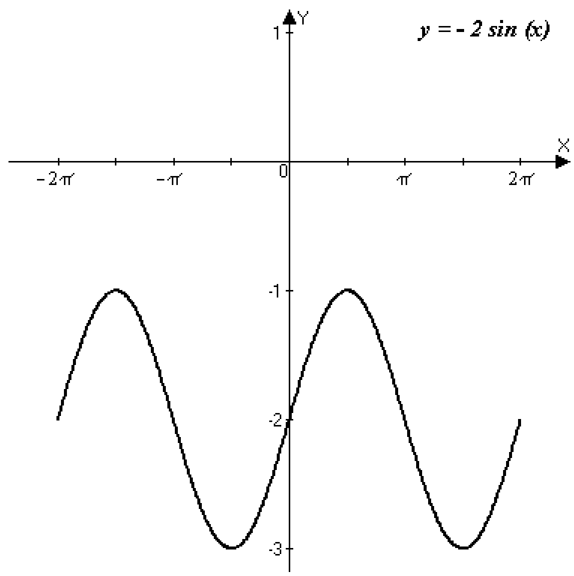
We next investigate another transformation of the function by multiplying the value of x by the constant B. In doing this, we are changing the period of the function, not its amplitude. In a normal cycle, the period of the function is equal to 2π radians, since that value of x represents one complete revolution of a circular point around the circumference of a circle, and beyond that the function begins to repeat itself. If we multiply x by the constant B, the period of the function now becomes equal to $2\pi / B$. For values of B greater than 1, the period is shortened, and for values of B less than 1, the period is lengthened. Some examples of the function $y = \sin Bx$ for different values of B are shown below.



A third transformation of the basic function is obtained by adding the constant C to the value of Bx. When we do this, we shift the entire function to the left or right, depending on whether C is positive or negative. If C is positive, the value of $Bx + C$ is greater than Bx, and the function has reached its position for a given value of x sooner than it otherwise would have. The pattern of the function has thus been shifted to the left to compensate for the smaller value of x required to produce a given y value. The amount of the shift is equal to C if B is equal to 1, but for other values B becomes a factor and the shift is equal to C / B. Because a positive value of C produces a shift to the left (a negative direction), the shift is said to be equal to $-C / B$. The shift is commonly referred to as a **phase shift**. Some examples of phase shifting are shown below.



A final transformation of the basic sine function is obtained by adding a constant *k* to the entire function. This simply translates the function vertically up or down by the value of *k*. An example of this translation is shown below.



We have thus discussed four different transformations of the basic sine function. A general form of the sine function which incorporates all of these possible transformations is $y = k + A \sin(Bx + C)$. Of course, if k and C are equal to 0 and A and B are equal to 1, we are left with the basic sine function $y = \sin x$ and there are no transformations.

We may do exactly the same transformations to the basic cosine function $y = \cos x$ as we have done to the sine function. As we noted in the previous section, the pattern of the cosine function is exactly the same as the sine function, except there is a horizontal shift in the function equal to one-fourth of the period. The general form of the cosine function is $y = k + A \cos (Bx + C)$.