Exponential Growth and Decay; Modeling Data

In this section, we will study some of the applications of exponential and logarithmic functions. Logarithms were invented by John Napier. Originally, they were used to eliminate tedious calculations involved in multiplying, dividing, and taking powers and roots of the large numbers that occur in different sciences. Computers and calculators have since eliminated the need of logarithms for these calculations. However, their need has not been removed completely. Logarithms arise in problems of exponential growth and decay because they are inverses of exponential functions. Because of the Laws of Logarithms, they also turn out to be useful in the measurement of the loudness of sounds, the intensity of earthquakes, and other processes that occur in nature.

Previously, we studied the formula for exponential growth, which models the growth of animal or bacteria population.

If n_0 is the initial size of a population experiencing exponential growth, then the population $n(t)$ at time t is modeled by the function

$$n(t) = n_0 e^{rt}$$

where r is the relative rate of growth expressed as a fraction of the population.

Now, we have the powerful logarithm, which will allow us to answer questions about the time at which a population reaches a certain size.

Example 1: Frog population projections

The frog population in a small pond grows exponentially. The current population is 85 frogs, and the relative growth rate is 18% per year.

(a) Find a function that models the number of frogs after t years.
(b) Find the projected population after 3 years.
(c) Find the number of years required for the frog population to reach 600.

Solution (a): To find the function that models population growth, we need to find the population $n(t)$. To do this, we use the formula for population growth with $n_0 = 85$ and $r = 0.18$. Then

$$n(t) = 85e^{0.18t}$$

Example 1 (Continued):

Solution (b): We use the population growth function found in (a) with $t = 3$.

$$n(3) = 85e^{0.18(3)}$$
$$n(3) \approx 145.86 \qquad \text{Use a calculator}$$

Thus, the number of frogs after 3 years is approximately 146.

Solution (c): Using the function we found in part (a) with $n(t) = 600$ and solving the resulting exponential equation for t, we get

$$
\begin{aligned}
85e^{0.18t} &= 600 \\
e^{0.18t} &\approx 7.059 && \text{Divide by 85} \\
\ln(e^{0.18t}) &\approx \ln(7.059) && \text{Take ln of each side} \\
0.18t &\approx \ln(7.059) && \text{Property of ln} \\
t &\approx \frac{\ln(7.059)}{0.18} && \text{Divide by 0.18} \\
t &\approx 10.86 && \text{Use a calculator}
\end{aligned}
$$

Thus, the frog population will reach 600 in approximately 10.9 years.

Example 2: Find the number of bacteria in a culture

A culture contains 10,000 bacteria initially. After an hour, the bacteria count is 25,000.

(a) Find the doubling period.
(b) Find the number of bacteria after 3 hours.

Solution (a): We need to find the function that models the population growth, $n(t)$. In order to find this, we must first find the rate r. To do this, we use the formula for population growth with $n_0 = 10{,}000$, $t = 1$, and $n(t) = 25{,}000$, and then solve for r.

$$
\begin{aligned}
10{,}000e^{r(1)} &= 25{,}000 \\
e^{r} &= 2.5 && \text{Divide by 10,000} \\
\ln(e^{r}) &= \ln(2.5) && \text{Take ln of each side} \\
r &= \ln(2.5) && \text{Property of ln} \\
r &\approx 0.91629 && \text{Use a calculator}
\end{aligned}
$$

Example 2 (Continued):

Now that we know $r \approx 0.91629$, we can write the function for the population growth:

$$n(t) = 10,000e^{0.91629t}$$

Recall that the original question is to find the doubling period, so we are not done yet. We need to find the time, t, when the population $n(t) = 20,000$. We use the population growth function found above and solve the resulting exponential equation for t.

$$\begin{aligned}
10,000e^{0.91629t} &= 20,000 \\
e^{0.91629t} &= 2 && \text{Divide by 10,000} \\
\ln(e^{0.91629t}) &= \ln(2) && \text{Take ln of each side} \\
0.91629t &= \ln(2) && \text{Property of ln} \\
t &= \frac{\ln(2)}{0.91629} && \text{Divide by 0.91629} \\
t &\approx 0.756 && \text{Use a calculator}
\end{aligned}$$

Thus, the bacteria count will double in about 0.75 hours.

Solution (b): Using the population growth function found in part (a), with rate $r = 0.91629$ and time $t = 3$, we find

$$\begin{aligned}
n(3) &= 10,000e^{0.91629(3)} \\
&\approx 156,249.66 && \text{Use a calculator}
\end{aligned}$$

So, the number of bacteria after 3 hours is about 156,250.

Radioactive Decay:

In radioactive substances the mass of the substance decreases, or decays, by spontaneously emitting radiation. The rate of decay is directly proportional to the mass of the substance. The amount of mass $m(t)$ remaining at any given time t can be shown to be modeled by the function

$$m(t) = m_0 e^{-rt}$$

where m_0 is the initial mass and r is the rate of decay. In general, physicists express the rate of decay in terms of **half-life**, the time required for half the mass to decay. Sometimes, we are given the half-life value and need to find the rate of decay. To obtain this rate, follow the next few steps. Let h represent the half-life and assume that our initial mass is 1 unit. This forces $m(t)$ to be ½ unit when $t = h$. Now, substituting all of this information into our model, we get

$$\frac{1}{2} = 1 \cdot e^{-rh}$$

$$\ln\left(\frac{1}{2}\right) = -rh \qquad \text{Take ln of each side}$$

$$r = -\frac{1}{h}\ln(2^{-1}) \qquad \text{Solve for r}$$

$$r = \frac{\ln(2)}{h} \qquad \ln(2^{-1}) = -\ln(2) \text{ by law 3}$$

This equation for r will allow us to find the rate of decay whenever we are given the half-life h.

If m_0 is the initial mass of a radioactive substance with half-life h, then the mass remaining at time t is modeled by the function:

$$m(t) = m_0 e^{-rt} \text{ where } r = \frac{\ln(2)}{h}.$$

Example 3: Radioactive Decay

The half-life of cesium-137 is 30 years. Suppose we have a 10 g sample.

(a) Find a function that models the mass remaining after t years.
(b) How much of the sample will remain after 80 years?
(c) After how long will only 2 g of the sample remain?
(d) Draw a graph of the sample mass as a function of time.

Solution (a): Using the model for radioactive decay with $m_0 = 10$ and $r = \left(\ln(2)/30\right) \approx 0.023105$, we have:

$$m(t) = 10e^{0.023105t}$$

Example 3 (Continued):

Solution (b): We use the function we found in part (a) with t = 80.

$$m(80) = 10e^{-0.023105(80)} \approx 1.575$$

Thus, approximately 1.6 g of cesium-137 remains after 80 years.

Solution (c): We use the function we found in part (a) with $m(t) = 2$ and solve the resulting exponential equation for t.

$$10e^{-0.023105t} = 2$$

$$e^{-0.023105t} = \frac{1}{5} \qquad \text{Divide by 10}$$

$$\ln(e^{-0.023105t}) = \ln\left(\frac{1}{5}\right) \qquad \text{Take ln of each side}$$

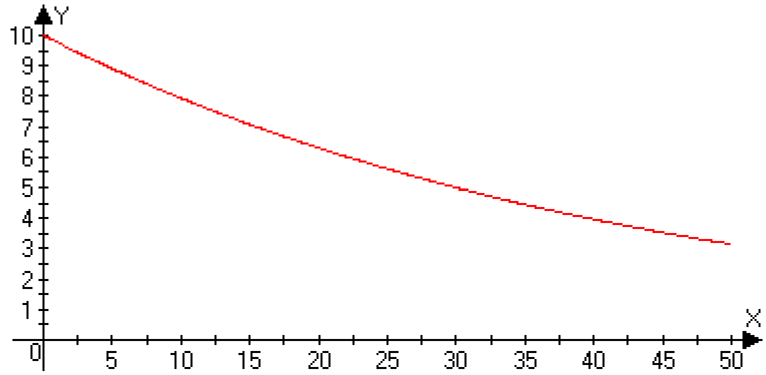
$$-0.023105t = \ln\left(\frac{1}{5}\right) \qquad \text{Property of ln}$$

$$t = -\frac{\ln\left(\frac{1}{5}\right)}{0.023105} \qquad \text{Divide by -0.023105}$$

$$t \approx 69.7 \qquad \text{Use a calculator}$$

The time required for the sample to decay to 2 g is about 70 years.

Solution (d): A graph of the function $m(t) = 10e^{-0.023105t}$ is shown below.



Newton's Law of Cooling:

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that the temperature difference is not too large. Using calculus, the following model can be deduced from this law.

If D_0 is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature T_S, then the temperature of the object at time t is modeled by the function

$$T(t) = T_S + D_0 e^{-kt}$$

where k is a positive constant that depends on the type of object.

Example 4: Newton's Law of Cooling is used in homicide investigations to determine the time of death. The normal body temperature is 98.6° F. Immediately following death, the body begins to cool. It has been determined experimentally that the constant in Newton's Law of Cooling is approximately $k = 0.1947$, assuming time is measured in hours. Suppose that the temperature of the surroundings is 60° F.

(a) Find a function $T(t)$ that models the temperature t hours after death.
(b) If the temperature of the body is now 72° F, how long ago was the time of death?
(c) Find the temperature of the body after 9 hours.

Solution (a):

The temperature of the surroundings is $T_S = 60°$ F, and the initial temperature difference is

$$D_0 = 98.6 - 60 = 38.6°\text{F}$$

So, by Newton's Law of Cooling and the given constant value $k = 0.1947$, the temperature after t hours is modeled by the function

$$T(t) = 60 + 38.6e^{-0.1947t}$$

Solution (b):

We use the function we found in part (a) with $T(t) = 72$ and solve the resulting exponential equation for t.

Example 4 (Continued):

$$\begin{aligned} 60+38.6e^{-0.1947t} &= 72 \\ 38.6e^{-0.1947t} &= 12 && \text{Subtract 60} \\ e^{-0.1947t} &\approx 0.31088 && \text{Divide by 38.6} \\ -0.1947t &\approx \ln(0.31088) && \text{Take ln of each side} \\ t &\approx -\frac{\ln(0.31088)}{0.1947} && \text{Divide by -0.1947} \\ t &\approx 6.0007 && \text{Use a calculator} \end{aligned}$$

Solution (c):

We use the function we found in part (a) with $t = 9$.

$$T(9) = 60 + 38.6e^{-0.1947(9)} \approx 66.69°\text{F}$$

Thus, the temperature of the body after 9 hours will be approximately 66.7° F.