Exponential and Logarithmic Equations

In this section, we solve equations that involve exponential or logarithmic equations. The techniques discussed here will be used in the next section for solving applied problems.

Exponential Equations:

An *exponential equation* is one in which the variable occurs in the exponent. For example,

$$3^x = 11$$

The variable x presents a difficulty because it is in the exponent. We can solve such an equation using the guidelines below.

Guidelines for Solving Exponential Equations:

1. Isolate the exponential expression on one side of the equation.

2. Take the logarithm of each side, then use the Laws of Logarithms to "bring down the exponent."

3. Solve for the variable.

Example 1: Find the solution of the exponential equation, correct to four decimal places.

(a) $3^x = 11$
(b) $e^{3x+4} - 6 = 11$
(c) $3^{x+2} = 5^x$
(d) $e^{2x} + e^x - 12 = 0$

Solution (a): To solve this equation we will use the guidelines for solving exponential equations given above.

Step 1: The first step in solving an exponential equation is to isolate the exponential expression on one side of the equation. Our equation $3^x = 11$ is already in this form so we can move on to the next step.

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Example 1 (Continued):

Step 2: The next step in solving an exponential equation is to take the logarithm of both sides, and then use the Laws of Logarithms to "bring down the exponent." Note that we use the common logarithm because our calculator can evaluate it, but we could have chosen to use any logarithm we like.

$$3^x = 11$$

$$\log 3^x = \log 11 \qquad \text{Take the logarithm of each side}$$

$$x \log 3 = \log 11 \qquad \text{Bring down the exponent}$$

Step 3: The final step in solving an exponential equation is the solve for the variable.

$$x \log 3 = \log 11$$

$$x = \frac{\log 11}{\log 3} \qquad \text{Divide both sides by } \log 3$$

$$x \approx 2.1827 \qquad \text{Use a calculator}$$

Step 4: We can check our answer by substituting x = 2.1827 into the original equation and using a calculator. We get

$$3^{2.1827} \approx 11 \quad \checkmark$$

Solution (b): Again we will follow the guidelines for solving exponential equations.

Step 1: Isolate the exponential expression on one side of the equation:

$$e^{3x+4} - 6 = 11$$

$$e^{3x+4} = 17 \qquad \text{Add 6 to both sides}$$

Step 2: Since the base of our exponential term is e, we choose to take the natural logarithm of both sides of the equation. Then use the Laws of Logarithms to "bring down the exponent."

$$e^{3x+4} = 17$$

$$\ln e^{3x+4} = \ln 17 \qquad \text{Take the natural logarithm of each side}$$

$$(3x+4)\ln e = \log 17 \qquad \text{Bring down the exponent}$$

Example 1 (Continued):

Step 3: Solve for the variable.

$$\begin{aligned} (3x+4)\ln e &= \ln 17 \\ 3x+4 &= \ln 17 && \ln e = 1 \\ 3x &= \ln 17 - 4 && \text{Subtract 4 from both sides} \\ x &= \frac{\ln 17 - 4}{3} && \text{Divide both sides by 3} \\ x &\approx -0.3889 && \text{Use a calculator} \end{aligned}$$

Step 4: Check the answer by substituting x = –0.8480 into the original equation and using a calculator.

$$e^{3(-0.3889)+4} \approx 17 \quad \checkmark$$

Solution (c): This problem is different from the previous two in that it has an exponential expression on both sides of the equation. We can still use the guidelines for solving exponential equations though.

Step 1: Our first step is to isolate the exponential expression on one side of the equation. Since our equation, $3^{x+2} = 5^x$, has two exponential expression, we want to make sure each expression is isolated on different sides of the equals sign. This is the case, and so we move on to the next step.

Step 2: Next we will take the logarithm of both sides. For this problem we have two exponents to "bring down" using the Laws of Logarithms.

$$\begin{aligned} 3^{x+2} &= 5^x \\ \log 3^{x+2} &= \log 5^x && \text{Take the logarithm of each side} \\ (x+2)\log 3 &= x\log 5 && \text{Bring down the exponents} \end{aligned}$$

Example 1 (Continued):

Step 3: Now we solve for the variable.

$$(x+2)\log 3 = x\log 5$$

$x\log 3 + 2\log 3 = x\log 5$ Distribute

$x\log 3 - x\log 5 = -2\log 3$ Rearrange terms

$x(\log 3 - \log 5) = -2\log 3$ Factor out x

$x = \dfrac{-2\log 3}{(\log 3 - \log 5)}$ Divide both sides by $(\log 3 - \log 5)$

$x \approx 4.3013$ Use a calculator

Step 4: Check the answer by substituting $x = -4.3013$ into the original equation and using a calculator.

$$3^{4.3013+2} \approx 5^{4.3013} \checkmark$$

Solution (d):

Step 1: In this problem our equation, $e^{2x} + e^{x} - 12 = 0$, is quadratic. We can isolate the exponential term by factoring.

$$e^{2x} + e^{x} - 12 = 0$$

$(e^{x})^{2} + e^{x} - 12 = 0$ Law of Exponents

$(e^{x} + 4)(e^{x} - 3) = 0$ Factor (a quadratic in e^{x})

$e^{x} + 4 = 0 \quad \text{or} \quad e^{x} - 3 = 0$ Zero-Product Property

$$e^{x} = -4 \qquad e^{x} = 3$$

Step 2: Since we now have two equations, we have a possibility of two solutions. We should perform the rest of our steps on each equation. Notice though that $e^{x} = -4$ has no solution because $e^{x} > 0$ for all x, so we can discard this equation. Now we will take the natural logarithm of both sides of $e^{x} = 3$, and use the Laws of Logarithms to "bring down the exponent."

$$e^{x} = 3$$

$\ln e^{x} = \ln 3$ Take the logarithm of each side

$x\ln e = \ln 3$ Bring down the exponent

Example 1 (Continued):

Step 3: Now we solve for the variable.

$$\begin{aligned} x\ln e &= \ln 3 \\ x &= \ln 3 && \color{red}{\ln e = 1} \\ x &\approx 1.0986 && \color{red}{\text{Use a calculator}} \end{aligned}$$

Step 4: Check the answer by substituting x = 1.0980 into the original equation and using a calculator. We get

$$e^{2(1.0986)} + e^{1.0986} - 12 \approx 0$$

Logarithmic Equations:

A *logarithmic equation* is one in which a logarithm of the variable occurs. For example

$$\log_7(x-3) = 17$$

We can solve this type of equation using the following guidelines.

Guidelines for Solving Logarithmic Equations:

1. Isolate the logarithmic term on one side of the equation; you may need to first combine the logarithmic terms.

2. Write the equation in exponential form (or raise the base to each side of the equation)

3. Solve for the variable.

Example 2: Solve the logarithmic equations for *x*.

(a) $\log_7(x-3) = 17$

(b) $10 + \ln(x+3) = 14$

(c) $\log_5(x-4) + \log_5(x+2) = 2$

(d) $\ln(x+2) = \ln(x)$

Example 2 (Continued):

Solution (a): To solve this equation we will use the guidelines for solving logarithmic equations given above.

Step 1: The first step in solving a logarithmic equation is to isolate the logarithmic term on one side of the equation. Our equation $\log_7 (x - 3) = 17$ is already in this form so we can move on to the next step.

Step 2: The next step in solving a logarithmic equation is to write the equation in exponential form, using the definition of the logarithmic function.

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

$$\log_7 (x-3) = 17 \quad \Leftrightarrow \quad 7^{17} = x - 3$$

Step 3: The final step in solving a logarithmic equation is the solve for the variable.

$$7^{17} = x - 3$$

$$x = 7^{17} + 3 \qquad \text{Add 3 to both sides}$$

Step 4: Check your answer. If $x = 7^{17} + 3$, we get

$$\log_7 \left(\left(7^{17} + 3 \right) - 3 \right) = \log_7 \left(7^{17} \right) \quad \checkmark$$

$$= 17$$

Solution (b): Again we will follow the guidelines for solving logarithmic equations.

Step 1: Isolate the logarithmic term on one side of the equation:

$$10 + \ln(x+3) = 14$$

$$\ln(x+3) = 4 \qquad \text{Subtract 10 from both sides}$$

Step 2: Write the equation in exponential form.

$$\ln(x+3) = 4$$

$$e^4 = x + 3 \qquad \text{Because } \ln \text{ is } \log_e$$

Example 2 (Continued):

Step 3: Solve for the variable.

$$e^4 = x + 3$$
$$x = e^4 - 3 \qquad \text{Subtract 3 from both sides}$$

Step 4: Check your answer. If $x = e^4 - 3$, we get

$$\begin{aligned} 10 + \ln\left(\left(e^4 - 3\right) + 3\right) &= 10 + \ln\left(e^4\right) \\ &= 10 + 4 \quad ✓ \\ &= 14 \end{aligned}$$

Solution (c): This problem is different from the previous two in that it has an exponential expression on both sides of the equation. We can still use the guidelines for solving exponential equations though.

Step 1: Our first step is to isolate the exponential expression on one side of the equation. Since our equation, $3^{x+2} = 5^x$, has two exponential expression, we want to make sure each expression is isolated on different sides of the equals sign. This is the case, and so we move on to the next step.

Step 2: Next we will take the logarithm of both sides. For this problem we have two exponents to "bring down" using the Laws of Logarithms.

$$3^{x+2} = 5^x$$
$$\log 3^{x+2} = \log 5^x \qquad \text{Take the logarithm of each side}$$
$$(x+2)\log 3 = x \log 5 \qquad \text{Bring down the exponents}$$

Example 2 (Continued):

Step 3: Now we solve for the variable.

$$
\begin{aligned}
(x+2)\log 3 &= x\log 5 \\
x\log 3 + 2\log 3 &= x\log 5 && \text{Distribute} \\
x\log 3 - x\log 5 &= -2\log 3 && \text{Rearrange terms} \\
x(\log 3 - \log 5) &= -2\log 3 && \text{Factor out } x \\
x &= \frac{-2\log 3}{(\log 3 - \log 5)} && \text{Divide both sides by } (\log 3 - \log 5) \\
x &\approx 4.3013 && \text{Use a calculator}
\end{aligned}
$$

Step 4: Check the answer by substituting $x = -4.3013$ into the original equation and using a calculator.

$$3^{4.3013+2} \approx 5^{4.3013} \qquad \checkmark$$

Solution (d):

Step 1: In this problem our equation, $e^{2x} + e^{x} - 12 = 0$, is quadratic. We can isolate the exponential term by factoring.

$$
\begin{aligned}
e^{2x} + e^{x} - 12 &= 0 \\
(e^{x})^{2} + e^{x} - 12 &= 0 && \text{Law of Exponents} \\
(e^{x}+4)(e^{x}-3) &= 0 && \text{Factor (a quadratic in } e^{x}\text{)} \\
e^{x}+4 = 0 \quad \text{or} \quad e^{x}-3 &= 0 && \text{Zero-Product Property} \\
e^{x} = -4 \qquad\qquad e^{x} &= 3
\end{aligned}
$$

Step 2: Since we now have two equations, we have a possibility of two solutions. We should perform the rest of our steps on each equation. Notice though that $e^{x} = -4$ has no solution because $e^{x} > 0$ for all x, so we can discard this equation. Now we will take the natural logarithm of both sides of $e^{x} = 3$, and use the Laws of Logarithms to "bring down the exponent."

$$
\begin{aligned}
e^{x} &= 3 \\
\ln e^{x} &= \ln 3 && \text{Take the logarithm of each side} \\
x\ln e &= \ln 3 && \text{Bring down the exponent}
\end{aligned}
$$

Example 2 (Continued):

Step 3: Now we solve for the variable.

$$x\ln e = \ln 3$$

$$x = \ln 3 \qquad \ln e = 1$$

$$x \approx 1.0986 \qquad \text{Use a calculator}$$

Step 4: Check the answer by substituting $x = 1.0980$ into the original equation and using a calculator. We get

$$e^{2(1.0986)} + e^{1.0986} - 12 \approx 0 \qquad ✓$$