# Properties of Logarithms

Since the exponential and logarithmic functions with base $a$ are inverse functions, the Laws of Exponents give rise to the Laws of Logarithms.

**Laws of Logarithms:** Let $a$ be a positive number, with $a \neq 1$. Let $A > 0$, $B > 0$, and $C$ be any real numbers.

| Law | Description |
|---|---|
| 1. $\log_a(AB) = \log_a A + \log_a B$ | The logarithm of a product of numbers is the sum of the logarithms of the numbers. |
| 2. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$ | The logarithm of a quotient of numbers is the difference of the logarithms of the numbers. |
| 3. $\log_a(A^C) = C\log_a A$ | The logarithm of a power of a number is the exponent times the logarithm of the number. |

**Example 1:** Use the Laws of Logarithms to rewrite the expression in a form with no logarithm of a product, quotient, or power.

(a) $\log_3\left(\frac{5x^3}{y^2}\right)$

(b) $\ln\sqrt[5]{x^2(z+1)}$

**Solution (a):**

$$\log_3\left(\frac{5x^3}{y^2}\right) = \log_3(5x^3) - \log_3 y^2 \qquad \text{Law 2}$$

$$= \log_3 5 + \log_3 x^3 - \log_3 y^2 \qquad \text{Law 1}$$

$$= \log_3 5 + 3\log_3 x - 2\log_3 y \qquad \text{Law 3}$$

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**Example 1 (Continued)**

**Solution (b):**

$$\begin{aligned}
\ln \sqrt[5]{x^2(z+1)} &= \ln\left(x^2(z+1)\right)^{\frac{1}{5}} \\
&= \tfrac{1}{5}\ln\left(x^2(z+1)\right) && \text{Law 3} \\
&= \tfrac{1}{5}\left[\ln x^2 + \ln(z+1)\right] && \text{Law 1} \\
&= \tfrac{1}{5}\ln x^2 + \tfrac{1}{5}\ln(z+1) && \text{Distribution} \\
&= \tfrac{1}{5}(2\ln x) + \tfrac{1}{5}\ln(z+1) && \text{Law 3} \\
&= \tfrac{2}{5}\ln x + \tfrac{1}{5}\ln(z+1) && \text{Multiplication}
\end{aligned}$$

**Example 2:** Rewrite the expression as a single logarithm.

(a) $3\log_4 5 + \log_4 10 - \log_4 7$

(b) $\log x - 2\log y + \tfrac{1}{2}\log\left(9z^2\right)$

**Solution (a):**

$$\begin{aligned}
3\log_4 5 + \log_4 10 - \log_4 7 &= \log_4 5^3 + \log_4 10 - \log_4 7 && \text{Law 3} \\
&= \log_4 125 + \log_4 10 - \log_4 7 && \text{Because } 5^3 = 125 \\
&= \log_4 (125 \cdot 10) - \log_4 7 && \text{Law 1} \\
&= \log_4 (1250) - \log_4 7 && \text{Simplification} \\
&= \log_4 \left(\frac{1250}{7}\right) && \text{Law 2}
\end{aligned}$$

**Example 2 (Continued):**

**Solution (b):**

$$
\begin{aligned}
\log x - 2\log y + \tfrac{1}{2}\log\left(9z^2\right) &= \log x - \log y^2 + \log\left(9z^2\right)^{\frac{1}{2}} && \text{Law 3} \\
&= \log x - \log y^2 + \log(3z) && \text{Simplification} \\
&= \log\left(\tfrac{x}{y^2}\right) + \log(3z) && \text{Law 2} \\
&= \log\left[\left(\tfrac{x}{y^2}\right)(3z)\right] && \text{Law 1} \\
&= \log\left(\tfrac{3zx}{y^2}\right) && \text{Simplification}
\end{aligned}
$$

**Example 3:** Evaluate the expression

(a) $\log_2 \sqrt[5]{16}$
(b) $\log_3 189 - \log_3 7$

**Solution (a):**

$$
\begin{aligned}
\log_2 \sqrt[5]{16} &= \log_2 16^{\frac{1}{5}} && \\
&= \tfrac{1}{5}\log_2 16 && \text{Law 3} \\
&= \tfrac{1}{5}\log_2 2^4 && \text{Because } 16 = 2^4 \\
&= \tfrac{1}{5}(4) && \text{Property 3 of Logarithms (Section 6.2)} \\
&= \tfrac{4}{5} && \text{Multiplication}
\end{aligned}
$$

**Solution (b):**

$$
\begin{aligned}
\log_3 189 - \log_3 7 &= \log_3 \tfrac{189}{7} && \text{Law 2} \\
&= \log_3 27 && \text{Simplification} \\
&= \log_3 3^3 && \text{Because } 27 = 3^3 \\
&= 3 && \text{Property 3 of Logarithms (Section 6.2)}
\end{aligned}
$$

**Change of Base:**

A calculator can be used to approximate the values of common logarithms (base 10) or natural logarithms (base *e*). However, sometimes we need to use logarithms to other bases. The following rule is used to convert logarithms from one base to another.

**Change of Base Formula:**

$$\log_b x = \frac{\log_a x}{\log_a b}$$

**Example 4:** Use the Change of Base Formula and a calculator to evaluate the logarithm, correct to six decimal places. Use either natural or common logarithms.

(a) $\log_6 17$

(b) $\log_5 2.33$

**Solution (a):**

The Change of Base Formula says

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

Thus, if we let the new base $a = 10$

$$\log_6 17 = \frac{\log_{10} 17}{\log_{10} 6} \approx 1.581246$$

**Solution (b):**

Again we will use the Change of Base Formula. This time we will let the new base be $a = e$.

$$\log_5 2.33 = \frac{\ln 2.33}{\ln 5} \approx 0.525568$$

**Example 5:** Simplify: $(\log_8 12)(\log_{12} 7)$

**Solution:**

Using the Change of Base Formula with the new base $a = 10$:

$$\log_8 12 = \frac{\log 12}{\log 8} \quad \text{and} \quad \log_{12} 7 = \frac{\log 7}{\log 12}$$

Thus,

$$\begin{aligned}(\log_8 12)(\log_{12} 7) &= \left(\frac{\log 12}{\log 8}\right)\left(\frac{\log 7}{\log 12}\right) \\ &= \frac{\log 7}{\log 8} && \text{Simplification} \\ &= \log_8 7 && \text{Change of Base Formula}\end{aligned}$$