Inverse Functions

Think about your math class—if we assume that there are no siblings in your class then we can make the following statement:

No two students in the class have the same set of parents.

This situation is similar to a one-to-one function.

Def: One-to-one functions:

A function within the domain A is called a <u>one-to-one function</u> if no two elements of A have the same image, that is,

$f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

To understand this definition better, let's go back to the situation in your math class. In this case, x_1 and x_2 would represent two students in your class, and $f(x_1)$ and $f(x_2)$ would represent the corresponding sets of parents. Obviously, if the two students are different people, then the corresponding sets of parents must be different also.

Another way to write this definition is the following:

If $f(x_1) = f(x_2)$ then $x_1 = x_2$

Horizontal line test:

One-to-one functions are very easy to identify graphically.

Def: Horizontal line test:

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

In other words, if you can draw at least one horizontal line *anywhere* on a graph, which intersects the graph more than once, then the graph is *not* one-to-one.

Consider the graph of a quadratic function. We already know that this is a parabola which opens upward or downward. You can find an infinite number of horizontal lines, which will intersect the graph more than once (twice in this case). Therefore, a quadratic function is *not* a one-to-one function.

Note: You may remember using the *vertical line test* previously in order to determine if a given graph is the graph of a function. The horizontal line test is different, and is used to determine if a given function is a one-to-one function.

Deciding whether a function is one-to-one:

It is very simple to tell whether a graph is the graph of a one-to-one function by using the horizontal line test. Now, we want to look at how to determine if a function is one-to-one just by looking at the function itself.

There is more than one way to prove that a function is or is not one-to-one. We will look at a couple of examples in order to help you begin to understand how to prove or determine whether a function is one-to-one.

Example 1: For the function $f(x) = x^4 + 5$, determine if the function is one-to-one.

Solution: Remember that there were two equivalent ways of stating the definition of a one-to-one function.

1) $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$
2) If $f(x_1) = f(x_2)$ then $x_1 = x_2$

So, we can use either statement. In this case, let's use the first statement.

Let $x_1 = -2$ and $x_1 = 2$. Using these values for our variables, the resulting functions should not be equal if the function is one-to-one (by definition).

$$\begin{aligned} f(x_1) &= f(-2) \\ &= (-2)^4 + 5 \\ &= 16 + 5 \\ &= 21 \end{aligned} \qquad \begin{aligned} f(x_2) &= f(2) \\ &= (2)^4 + 5 \\ &= 16 + 5 \\ &= 21 \end{aligned}$$

Since $f(x_1) = f(x_2)$, then we do not have a one-to-one function.

Example 2: Show that the function $f(x) = 6 - 3x$ is one-to-one.

Solution: Let's use statement (2) from above (If $f(x_1) = f(x_2)$ then $x_1 = x_2$) for this problem.

Suppose there are two numbers x_1 and x_2 such that $f(x_1) = f(x_2)$.

$$\begin{aligned} 6 - 3x_1 &= 6 - 3x_2 \\ -3x_1 &= -3x_2 \\ x_1 &= x_2 \end{aligned}$$

Therefore, f is one-to-one.

The inverse of a function:

It is natural to look at inverse functions along with one-to-one functions, because all one-to-one functions have inverse functions.

To see how this works, let's revisit the situation from the beginning of this section. We want to look again at the students in your math class. Now, this will require us to make an assumption in order to make the analogy work correctly, but I think it will help to give us a good word picture. Let us assume that every set of parents represented has only one child (being the one in the math class). If you remember from before, we said that every child corresponded to one unique set of parents, meaning that no two students in the class have the same set of parents. The reverse can also be said, now. Each set of parents in the group has only one student in the classroom (and of course, no two sets of parents have the same student). If we were talking in terms of functions, this would be representative of the *inverse* of the "function" above.

Def: The inverse of a function:

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y \quad \text{for any } y \text{ in } B.$$

(This definition is much easier understood by looking at the diagrams in your book, so I would encourage you to study those figures.)

The easiest way to work with these types of functions is to look at specific values. We will start with an example using specific values in order to feel more comfortable working with inverse functions.

Example 3:

(a) If $f(3) = 15$, find $f^{-1}(15)$.

If we compare this to the definition, we can see that $x = 3$ and $y = 15$. Therefore, $f^{-1}(15) = 3$.

(b) If $f(-3) = 7$ find $f^{-1}(7)$.

By definition,
$f^{-1}(7) = -3$ since $f(-3) = 7$.

When working with the actual function (instead of specific values) it is necessary to follow a simple three-step process in order to determine the inverse of a function. We'll look at an example to help us see this process.

Example 4: Find the inverse of the function $f(x) = 5x + 7$.

Step 1: Write $y = f(x)$.

We substitute the expression in for $f(x)$.

$y = 5x + 7$

Step 2: Solve the resulting equation for x in terms of y (if possible).

$$y = 5x + 7$$
$$y - 7 = 5x$$
$$\frac{y-7}{5} = x$$

so $x = \frac{y-7}{5}$

Step 3: Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

$y = \frac{x-7}{5}$ (Interchange x and y.)

Therefore, the inverse function is $f^{-1}(x) = \frac{x-7}{5}$.

Example 5: Find the inverse of the function $f(x) = \dfrac{2x-1}{x+1}$.

Step **1:** $y = \dfrac{2x-1}{x+1}$

Step 2: (Solve for *x* in terms of *y*.)

$$y = \frac{2x-1}{x+1}$$
$$y(x+1) = 2x-1$$
$$xy + y = 2x-1$$
$$y+1 = 2x - xy$$
$$y+1 = x(2-y)$$
$$\frac{y+1}{2-y} = x$$

So $x = \dfrac{y+1}{2-y}$

Step 3: Interchanging *x* and *y* we have:

$$y = \frac{x+1}{2-x}$$

Therefore, $f^{-1}(x) = \dfrac{x+1}{2-x}$.

Checking your answers using the Inverse Function Property:

It is very helpful to note an important property of inverse functions.

Let *f* be a one-to-one function with domain *A* and range *B*. The inverse function f^{-1} satisfies the following cancellation properties.

$f^{-1}(f(x)) = x$ for every *x* in *A*
$f(f^{-1}(x)) = x$ for every *x* in *B*

Conversely, any function f^{-1} satisfying these equations is the inverse of *f*.

This property can be used to check your answers.

Consider **Example 4** from above.

Check: According to the Inverse Function Property, $f^{-1}(5x + 7) = x$ should be a true statement.

(i) So, we have

$$f^{-1}(5x+7) = \frac{(5x+7)-7}{5}$$
$$= \frac{5x}{5}$$
$$= x$$

(ii) Also, $f(\frac{x-7}{5}) = x$ should be a true statement.

$$f\left(\frac{x-7}{5}\right) = 5\left(\frac{x-7}{5}\right) + 7$$
$$= (x-7) + 7$$
$$= x$$

So, by the Property of Inverse Function we can see that f and f^{-1} are inverses of each other.