More on Functions and Their Graphs

Difference Quotient

$\dfrac{f(a+h)-f(a)}{h}$ is known as the difference quotient and is used exclusively with functions. The objective to keep in mind is to factor the *h* appearing in the denominator from the problem, as seen in the following examples.

Example 1: Find the difference quotient of the function.

a.) $\mathbf{f(x) = 2x^2 + 3x + 4}$

Step 1. Find f(a) and f(a + h).

$f(a + h) = 2(a + h)^2 + 3(a + h) + 4$

$f(a + h) = 2(a^2 + 2ah + h^2) + 3a + 3h + 4$

$f(a + h) = 2a^2 + 4ah + 2h^2 + 3a + 3h + 4$

$f(a) = 2a^2 + 3a + 4$

Step 2. Substitute into the difference quotient.

$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{2a^2+4ah+2h^2+3a+3h+4-[2a^2+3a+4]}{h}\\ &= \frac{2a^2+4ah+2h^2+3a+3h+4-2a^2-3a-4}{h}\\ &= \frac{4ah+2h^2+3h}{h}\\ &= \frac{h(4a+2h+3)}{h}\\ &= 4a+2h+3\end{aligned}$$

Example 1 (Continued):

b.) $f\left(x\right)=\dfrac{2}{x+1}$

Step 1. Find f(a) and f(a+h).

$$f\left(a+h\right)=\frac{2}{\left(a+h\right)+1}=\frac{2}{a+h+1}$$

$$f\left(a\right)=\frac{2}{\left(a\right)+1}=\frac{2}{a+1}$$

Step 2. Substitute into the difference quotient.

$$\begin{aligned}
\frac{f\left(a+h\right)-f\left(a\right)}{h} &= \frac{\left(\dfrac{2}{a+h+1}\right)-\left(\dfrac{2}{a+1}\right)}{h} \\
&= \frac{\left(\dfrac{a+1}{a+1}\right)\left(\dfrac{2}{a+h+1}\right)-\left(\dfrac{2}{a+1}\right)\left(\dfrac{a+h+1}{a+h+1}\right)}{h} \\
&= \frac{\left(\dfrac{2a+2}{a^2+ah+a+a+h+1}\right)-\left(\dfrac{2a+2h+2}{a^2+ah+a+a+h+1}\right)}{h} \\
&= \frac{\dfrac{2a+2-\left[2a+2h+2\right]}{a^2+ah+2a+h+1}}{h} \\
&= \frac{\dfrac{2a+2-2a-2h-2}{a^2+ah+2a+h+1}}{h} \\
&= \frac{\dfrac{-2h}{a^2+ah+2a+h+1}}{h} \\
&= \frac{-2h}{a^2+ah+2a+h+1}\times\frac{1}{h} \\
&= \frac{-2}{a^2+ah+2a+h+1}
\end{aligned}$$

Piecewise Defined Functions:

The last type of function we will consider is called a piecewise defined function. It is not one of the basic functions we have already looked at, so the graphs will not follow any particular shape. Each piecewise defined function is different, but some general guidelines can be used to graph these functions.

By *piecewise defined*, we mean that the function is defined by a different expression for different values of the dependent variable. The graph will have different curves, or "pieces" coinciding with the different ways that the function is defined. With these types of functions, it is necessary to look at each piece of the function separately, then graph.

Example 2: Graph the following function:

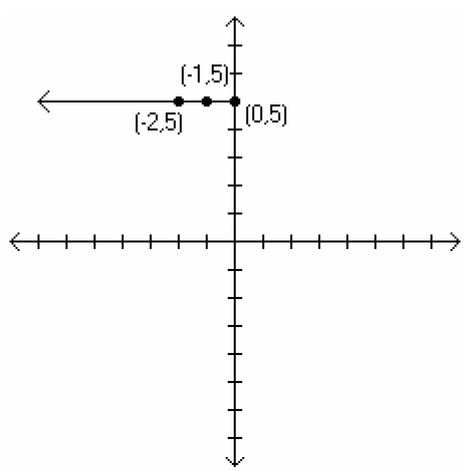
$$f(x) = \begin{cases} 5 & if \quad x \le 0 \\ |x| & if \quad 0 < x < 3 \\ x+1 & if \quad 3 \le x \end{cases}$$

Solution: We have three pieces of the graph to work with.

Step 1: First, we have the segment where x $\le$ 0 or $(\infty, 0]$ in interval notation.

x	$f(x) = 5$
0	5
-1	5
-2	5

Graphing this portion of the function, we have:



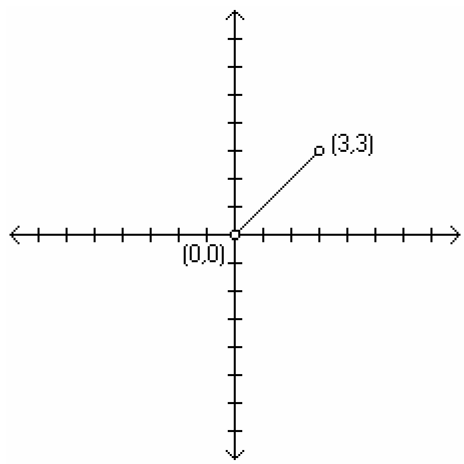
Notice, we have a solid circle at $x = 0$, since the condition for this piece of the graph is for x to be less than *or equal to* zero.

Example 2 (Continued):

Step 2: Graphing the next portion of the graph where $0 < x < 3$ is defined on the interval (0, 3), we have the following table:

x	$f(x) = \lvert x \rvert$
0	0
1	1
3	3

Graphing this portion of the function, we have:



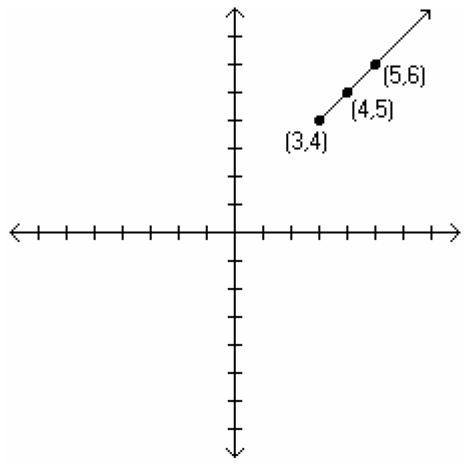
Notice that we have open circles at the values where $x = 0$ and $x = 3$, because this point is excluded from the graph based on the given conditions.

Step 3: The final portion of the graph is where $x \geq 3$ is defined to be $[3, \infty)$, so we can make the following table:

x	$f(x) = x + 1$
3	4
4	5
5	6

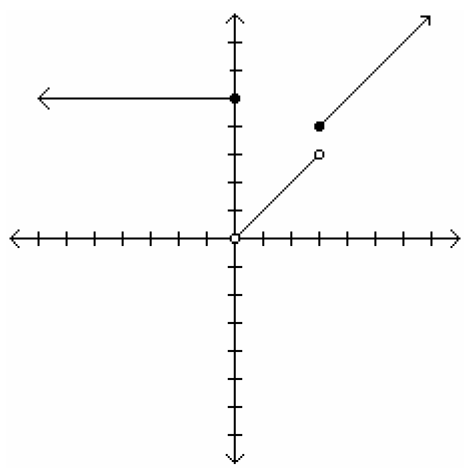
Example 2 (Continued):

So we have,



We use a solid circle to show that the point where $x = 3$ is included in this portion of the graph.

Step 4: Now we have the three pieces of the graph, but in order to correctly graph the whole function, we must graph all three pieces.



There are many different types of piecewise defined functions. Some have patterns like stair steps, and others have less of a pattern (as in the above example). In each case, you can follow the general steps just shown. Begin by breaking down the function into the separate pieces according to the conditions given for the independent variable (x in this case). Then you can graph each piece. Remember to pay close attention to the signs $<$, $>$, $\leq$, and $\geq$. This will determine whether you have an open or closed dot at the end points of the graph.

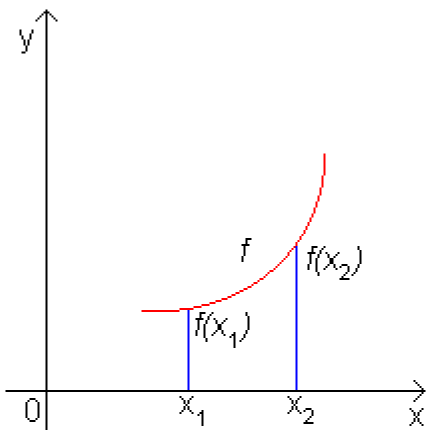
Increasing and Decreasing Functions:

Functions that are used to model real-life scenarios are usually not completely constant. There will be periods (or intervals) where they increase or decrease. For example, if you set your cruise control at 70mph in your car, there will be times when the speed falls slightly below or rises slightly above 70mph (such as when you go up or down a hill). In these instances, the function of the rate would be decreasing or increasing.

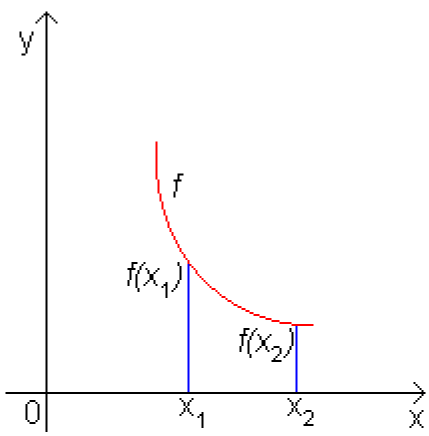
This can be easily seen when the function is graphed. When the graph rises on an interval, the function is increasing. When the graph falls on an interval, the function is decreasing.

Def: Increasing and Decreasing Functions:

f is **increasing** on an interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$

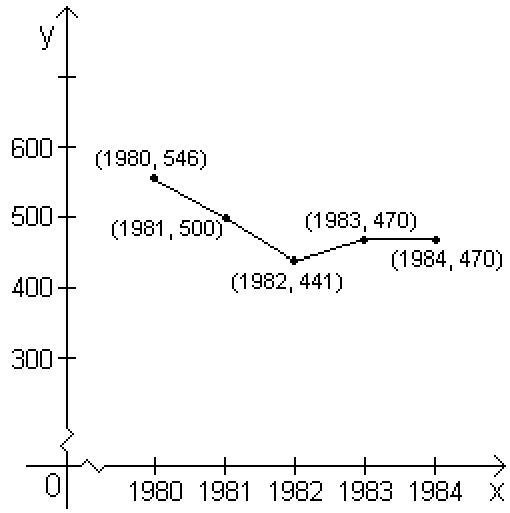


f is **decreasing** on an interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$



Example 3:

For a particular small town in west Texas, the population was noted over a five year period of time. The values are recorded on the following graph.



(a) For what period of time was the population increasing?
(b) For what period of time was the population decreasing?
(c) For what period of time was the population constant (i.e. no change in population)?

Solution:

(a) We can see from the graph that the population was increasing from 1982 to 1983.

(b) We can see from the graph that the population was decreasing from 1980 to 1982.

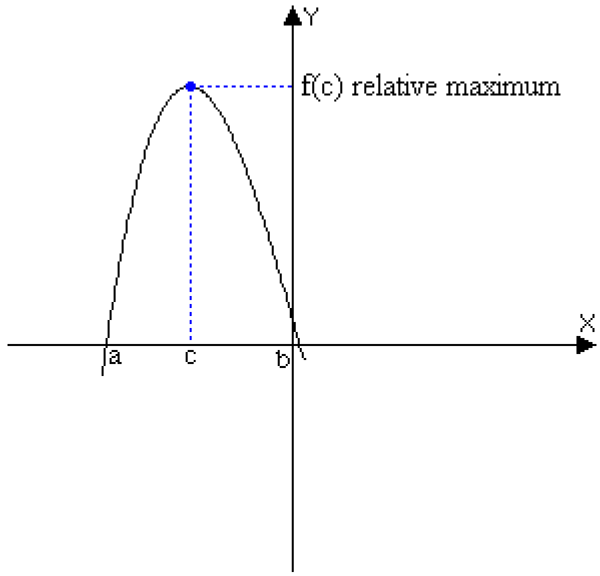
(c) The population was constant from 1983 to 1984.

Relative Extrema

The relative extrema of a function are the points where a relative (local) maximum or minimum point exists in the open interval (a, b). When locating the relative extrema you will want to look at the critical numbers derived from the first derivative and any endpoints of the function.

A relative maximum would be the highest point in the open interval (a, b). Therefore, the value of the function at the relative maximum point should be greater than the value of the function at all other points in the same open interval.

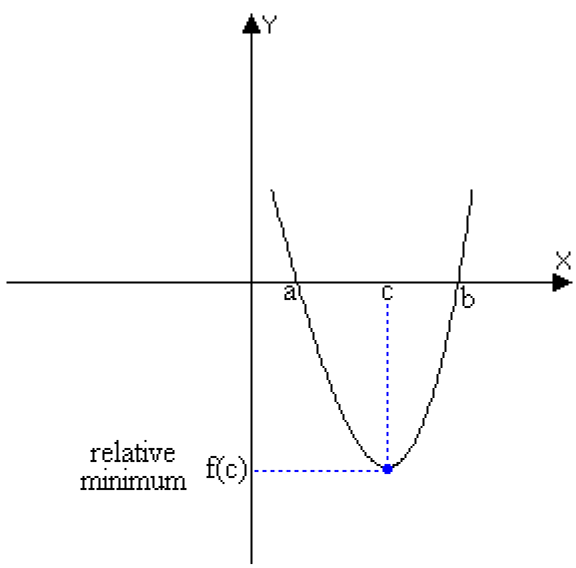
> If c is a number located in the open interval of (a, b) and is included in the domain of the function f, then a relative maximum exists at f(c) when
>
> $f(x) \le f(c)$ for all x in the open interval (a, b)



A relative minimum on the other hand would be the lowest point in the open interval (a, b).

If c is a number located in the open interval of (a, b) and is included in the domain of the function f, then a relative minimum exists at f(c) when

$f(x) \geq f(c)$ for all x in the open interval (a, b)



Even and Odd Functions:

A function f is an **even function** if $f(-x) = f(x)$ for all x in the domain of f. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x).$$

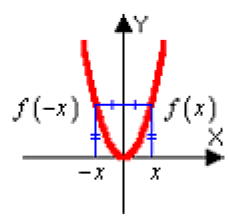
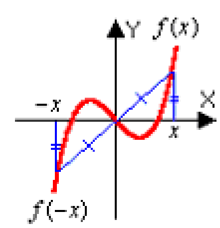
If a function f is even, then the graph of f is symmetric with respect to the y-axis. This means that if we reflect the graph of f in the y-axis we will obtain the same graph.

A function f is an **odd function** if $f(-x) = -f(x)$ for all x in the domain of f. For instance, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

The graph of an odd function is symmetric about the origin. This means that if we reflect the graph in the y-axis, and then reflect it in the x-axis we will obtain the same graph.

Even and Odd Functions

Definition	Symmetry of Graph of f	What the Graph Looks Like
f is even if $f(-x) = f(x)$ for all x in the domain of f	Graph of f is symmetric with respect to the y-axis	
f is odd if $f(-x) = -f(x)$ for all x in the domain of f	Graph of f is symmetric with respect to the x-axis	

Example 4: Determine whether the functions are even, odd, or neither.

(a) $g(x) = -3x^7 + 7x^3 + x$ (b) $f(x) = x^6 + 9x^2 - 100$

(c) $h(x) = 10x^3 - 2x^2$

Solution (a):

Step 1: First we will determine what $g(-x)$ is. We do this by substituting $-x$ for each x in $g(x) = -3x^7 + 7x^3 + x$ and simplifying. Therefore

$$\begin{aligned} g(-x) &= -3(-x)^7 + 7(-x)^3 + (-x) \\ &= 3x^7 - 7x^3 - x \\ &= -\left(-3x^7 + 7x^3 + x\right) \\ &= -g(x) \end{aligned}$$

Step 2: Since $g(-x) \neq g(x)$, the function $g(x) = -3x^7 + 7x^3 + x$ is not even.

Step 3: Since $g(-x) = -g(x)$, the function $g(x) = -3x^7 + 7x^3 + x$ is odd.

Solution (b):

Step 1: First we will determine what $f(-x)$ is. Substitute $-x$ for each x in $f(x) = x^6 + 9x^2 - 100$ and simplify.

$$\begin{aligned} f(-x) &= (-x)^6 + 9(-x)^2 - 100 \\ &= x^6 + 9x^2 - 100 \\ &= f(x) \end{aligned}$$

Step 2: Since $f(-x) = f(x)$, the function $f(x) = x^6 + 9x^2 - 100$ is even.

Step 3: Since $f(x) = x^6 + 9x^2 - 100$ is even, we do not need to test whether the function is odd.

note: The only function that is both even and odd is $f(x) = 0$.

Example 4 (Continued):

Solution (c):

Step 1: Determine what $h(-x)$ is.

$$\begin{aligned} h(-x) &= 10(-x)^3 - 2(-x)^2 \\ &= -10x^3 - 2x^2 \end{aligned}$$

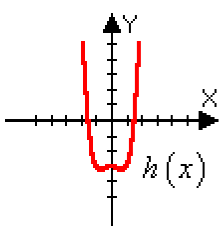
Step 2: Since $h(-x) \neq h(x)$, the function $h(x) = 10x^3 - 2x^2$ is not even.

Step 3: Since $h(-x) \neq -h(x)$, the function $h(x) = 10x^3 - 2x^2$ is not odd.

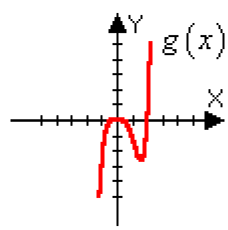
Step 4: Therefore the function $h(x) = 10x^3 - 2x^2$ is neither even nor odd.

Example 5: Based on the graphs of the functions, determine whether the functions are even, odd, or neither.

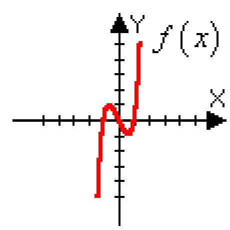
(a)



(b)



(c)



Solution (a): We determine whether a graph is even, odd or neither by checking its symmetry. The graph of $h(x)$ is symmetric to the y-axis, therefore $h(x)$ is an even function.

Solution (b): The graph of $g(x)$ is neither symmetric to the y-axis nor to the origin, therefore $g(x)$ is neither an even nor odd function.

Solution (c): The graph of $f(x)$ is symmetric to the origin, therefore $f(x)$ is an odd function.