Transformations of Functions

An alternative way to graphing a function by plotting individual points is to perform **transformations** to the graph of a function you already know.

Library Functions:

In previous sections, we learned the graphs of some basic functions. Collectively, these are known as the graphs of the **library functions**.

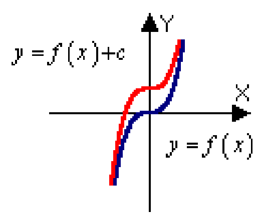
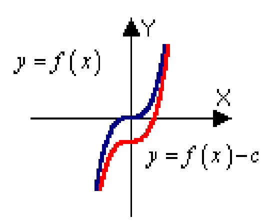
These are the graphs of the functions we will begin to perform transformations on to find the graphs of other functions.

We will discuss three types of transformations: shifting, reflecting, and stretching/shrinking.

Vertical Shifting:

Adding a constant to a function will shift its graph vertically (i.e. $y = f(x) + c$). Adding a positive constant c will shift the graph upward c units, while adding a negative constant c will shift it downward c units.

Vertical Shifts of Graphs

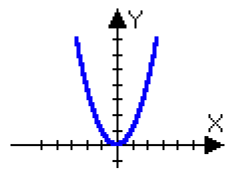
Equation	How to Obtain the Graph	What the Graph Looks Like
$y = f(x) + c$ $(c > 0)$	Shift the graph of $y = f(x)$ upward c units.	
$y = f(x) - c$ $(c > 0)$	Shift the graph of $y = f(x)$ downward c units.	

Example 1: Sketch the graph of each function.

(a) $h(x) = x^2 + 2$ (b) $g(x) = |x| - 3$

Solution (a):

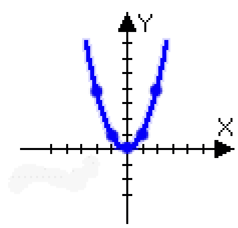
Step 1: First, we determine which library function best matches our given function. Since our function has an x^2 in it, we will use the library function $f(x) = x^2$.



$f(x) = x^2$

Step 2: Now that we know the library function we will be using, we need a set of points from the graph of $f(x) = x^2$ to work with. We can choose any points from the graph, but let's choose some easy ones.

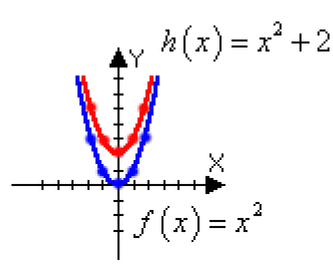
(x,y)
$(-2,4)$
$(-1,1)$
$(0,0)$
$(1,1)$
$(2,4)$



$f(x) = x^2$

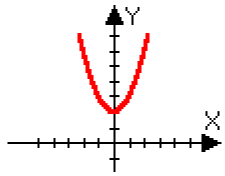
Step 3: The function $h(x) = x^2 + 2$ is of the form $y = f(x) + c$, so we know the graph of $h(x)$ will be the same as that of $f(x)$, but shifted upward 2 units. Thus, we can obtain points on the graph of $h(x)$ by taking our points from the graph of $f(x) = x^2$ and adding 2 to each of the y-values.

$f(x)$	$h(x)$
$(-2,4)$	$(-2,6)$
$(-1,1)$	$(-1,3)$
$(0,0)$	$(0,2)$
$(1,1)$	$(1,3)$
$(2,4)$	$(2,6)$



Example 1 (Continued):

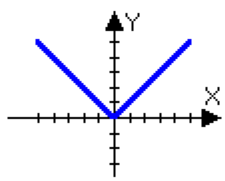
Step 4: Thus we have obtained the graph of $h(x) = x^2 + 2$ by transforming the graph of $f(x) = x^2$.



$h(x) = x^2 + 2$

Solution (b):

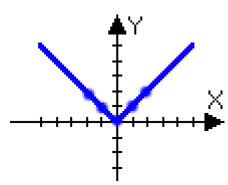
Step 1: First, we determine which library function best matches our given function. Since our function has an $|x|$ in it, we will use the library function $f(x) = |x|$.



$f(x) = |x|$

Step 2: Now we choose a set of points from the graph of $f(x) = |x|$ to work with.

(x, y)
$(-2, 2)$
$(-1, 1)$
$(0, 0)$
$(1, 1)$
$(2, 2)$

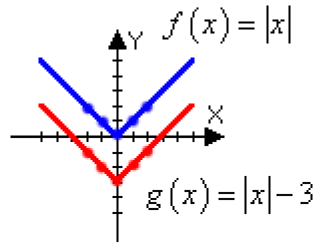


$f(x) = |x|$

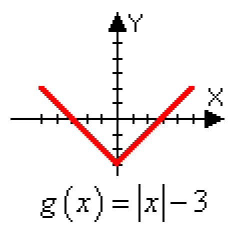
Example 1 (Continued):

Step 3: The function $g(x) = |x| - 3$ is of the form $y = f(x) - c$, so we know the graph of $g(x)$ will be the same as that of $f(x)$, but shifted downward 3 units. Thus, we can obtain points on the graph of $g(x)$ by taking our points from the graph of $f(x) = |x|$ and subtracting 3 from each of the y-values.

$f(x)$	$g(x)$
$(-2, 2)$	$(-2, -1)$
$(-1, 1)$	$(-1, -2)$
$(0, 0)$	$(0, -3)$
$(1, 1)$	$(1, -2)$
$(2, 2)$	$(2, -1)$



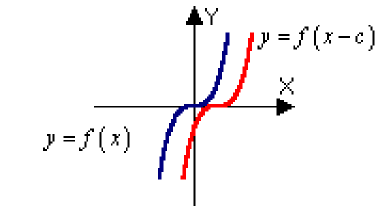
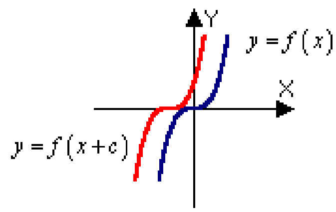
Step 4: Thus we have obtained the graph of $g(x) = |x| - 3$ by transforming the graph of $f(x) = |x|$.



Horizontal Shifting:

A horizontal shift is represented in either the form $y = f(x - c)$ or $y = f(x + c)$. Suppose we know the graph of $y = f(x)$. The value of $f(x - c)$ at x is the same as the value of $f(x)$ at $x - c$. Since $x - c$ is c units to the left of x, it follows that the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted to the right c units. Likewise, the graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted to the left c units.

Horizontal Shifts of Graphs

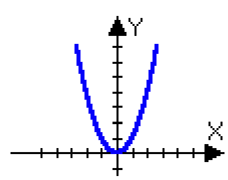
Equation	How to Obtain the Graph	What the Graph Looks Like
$y = f(x - c)$ $(c > 0)$	Shift the graph of $y = f(x)$ to the right c units.	
$y = f(x + c)$ $(c > 0)$	Shift the graph of $y = f(x)$ to the left c units.	

Example 2: Sketch the graph of each function.

(a) $h(x) = (x + 3)^2$ (b) $g(x) = \sqrt{x - 4}$

Solution (a):

Step 1: First, we determine which library function best matches our given function. Since our function has an $(x + 3)^2$ in it, we will use the library function $f(x) = x^2$.

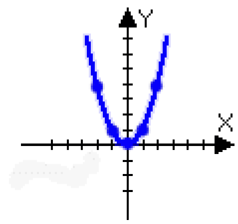


$f(x) = x^2$

Example 2 (Continued):

Step 2: Now we choose a set of points from the graph of $f(x) = x^2$ to work with.

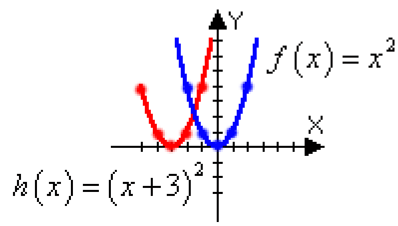
(x, y)
$(-2, 4)$
$(-1, 1)$
$(0, 0)$
$(1, 1)$
$(2, 4)$



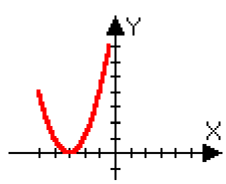
$f(x) = x^2$

Step 3: The function $h(x) = (x + 3)^2$ is of the form $y = f(x + c)$, so we know the graph of $h(x)$ will be the same as that of $f(x)$, but shifted left 3 units. Thus, we can obtain points on the graph of $h(x)$ by taking our points from the graph of $f(x) = x^2$ and subtracting 3 from each of the x-values.

$f(x)$	$h(x)$
$(-2, 4)$	$(-5, 4)$
$(-1, 1)$	$(-4, 1)$
$(0, 0)$	$(-3, 0)$
$(1, 1)$	$(-2, 1)$
$(2, 4)$	$(-1, 4)$



Step 4: Thus we have obtained the graph of $h(x) = (x + 3)^2$ by transforming the graph of $f(x) = x^2$.

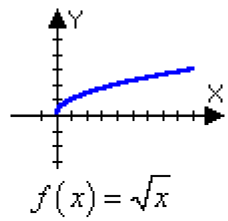


$h(x) = (x+3)^2$

Example 2 (Continued):

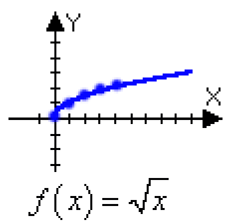
Solution (b):

Step 1: First, we determine which library function best matches our given function. Since our function has an $\sqrt{x-4}$ in it, we will use the library function $f(x) = \sqrt{x}$.



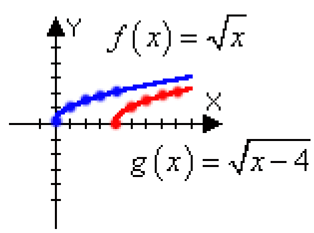
Step 2: Now we choose a set of points from the graph of $f(x) = \sqrt{x}$ to work with.

(x, y)
$(0,0)$
$(1,1)$
$(2,\sqrt{2})$
$(3,\sqrt{3})$
$(4,2)$



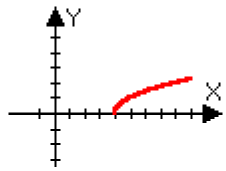
Step 3: The function $g(x) = \sqrt{x-4}$ is of the form $y = f(x - c)$, so we know the graph of $g(x)$ will be the same as that of $f(x)$, but shifted right 4 units. Thus, we can obtain points on the graph of $g(x)$ by taking our points from the graph of $f(x) = \sqrt{x}$ and adding 4 to each of the x-values.

$f(x)$	$g(x)$
$(0,0)$	$(0,4)$
$(1,1)$	$(1,5)$
$(2,\sqrt{2})$	$(2,\sqrt{2}+4)$
$(3,\sqrt{3})$	$(3,\sqrt{3}+4)$
$(4,2)$	$(4,6)$



Example 2 (Continued):

Step 4: Thus we have obtained the graph of $g(x) = \sqrt{x-4}$ by transforming the graph of $f(x) = \sqrt{x}$.

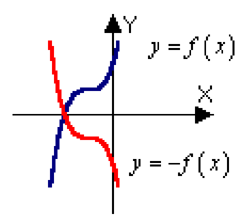
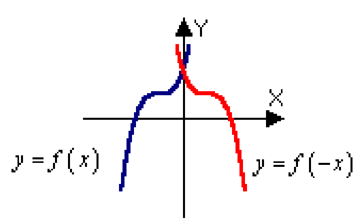


$g(x) = \sqrt{x-4}$

Reflecting Graphs:

If we know the graph of $y = f(x)$, we can obtain the graph of its reflection in the *x*-axis by multiplying the *y*-coordinate of each point on the graph of $y = f(x)$ by –1. This would give the graph of $y = -f(x)$. Similarly, the value of $y = f(-x)$ at *x* is the same as the value of $y = f(x)$ at $-x$, and so the graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ in the *y*-axis.

Reflecting Graphs

Equation	How to Obtain the Graph	What the Graph Looks Like
$y = -f(x)$	Reflect the graph of $y = f(x)$ in the *x*-axis.	
$y = f(-x)$	Reflect the graph of $y = f(x)$ in the *y*-axis.	

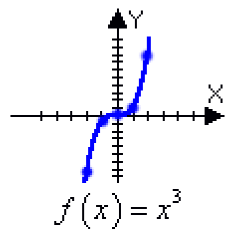
Example 3: Sketch the graph of each function.

(a) $h(x) = -x^3$ (b) $g(x) = \sqrt[3]{-x}$

Solution (a):

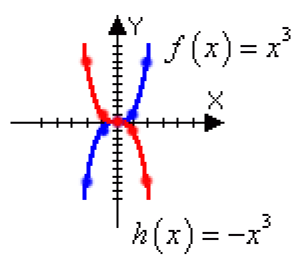
Step 1: Since our function has an x^3 in it, we will start with the library function $f(x) = x^3$. Choose a set of points from the graph of $f(x) = x^3$ to work with.

(x, y)
$(-2, -8)$
$(-1, -1)$
$(0, 0)$
$(1, 1)$
$(2, 8)$

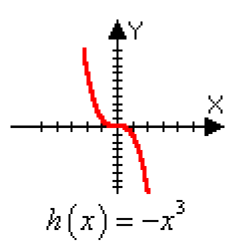


Step 2: The function $h(x) = -x^3$ is of the form $y = -f(x)$, so we know the graph of $h(x)$ will be the same as that of $f(x)$, but reflected in the x-axis. Thus, we can obtain points on the graph of $h(x)$ by multiplying the y-coordinate of each point from the graph of $f(x)$ by –1.

$f(x)$	$h(x)$
$(-2, -8)$	$(-2, 8)$
$(-1, -1)$	$(-1, 1)$
$(0, 0)$	$(0, 0)$
$(1, 1)$	$(1, -1)$
$(2, 8)$	$(2, -8)$



Step 3: Thus we have obtained the graph of $h(x) = -x^3$ by transforming the graph of $f(x) = x^3$.

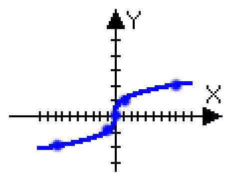


Example 3 (Continued):

Solution (b):

Step 1: Since our function has an $\sqrt[3]{-x}$ in it, we will start with the library function $f(x)=\sqrt[3]{x}$. Choose a set of points from the graph of $f(x)=\sqrt[3]{x}$ to work with.

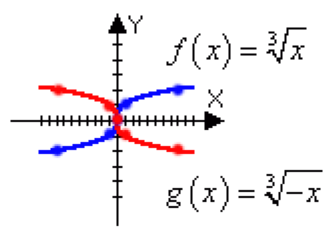
(x,y)
$(-8,-2)$
$(-1,-1)$
$(0,0)$
$(1,1)$
$(8,2)$



$f(x)=\sqrt[3]{x}$

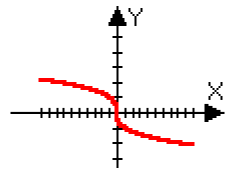
Step 2: The function $g(x)=\sqrt[3]{-x}$ is of the form $y=f(-x)$, so we know the graph of $g(x)$ will be the same as that of $f(x)$, but reflected in the *y*-axis. Thus, we can obtain points on the graph of $g(x)$ by taking our points from the graph of $f(x)=\sqrt[3]{x}$ and multiplying each of the *x*-values by -1.

$f(x)$	$g(x)$
$(-8,-2)$	$(8,-2)$
$(-1,-1)$	$(1,-1)$
$(0,0)$	$(0,0)$
$(1,1)$	$(-1,1)$
$(8,2)$	$(-8,2)$



Example 3 (Continued):

Step 3: Thus we have obtained the graph of $g(x) = \sqrt[3]{-x}$ by transforming the graph of $f(x) = \sqrt[3]{x}$.

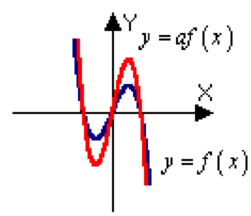
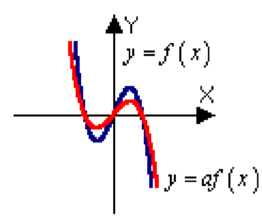


$g(x) = \sqrt[3]{-x}$

Vertical Stretching and Shrinking:

If we multiply a function by a constant a (i.e. $y = a\,f(x)$), the graph of the function is either vertically stretched if $a > 1$, or vertically shrunken if $0 < a < 1$. The y-coordinate of $y = a\,f(x)$ at x is the same as the y-coordinate of $y = f(x)$ multiplied by a.

Vertical Stretching and Shrinking of Graphs

Equation	How to Obtain the Graph	What the Graph Looks Like
$y = af(x)$ $(a > 1)$	Stretch the graph of $y = f(x)$ vertically by a factor of a.	
$y = af(x)$ $(0 < a < 1)$	Shrink the graph of $y = f(x)$ vertically by a factor of a.	

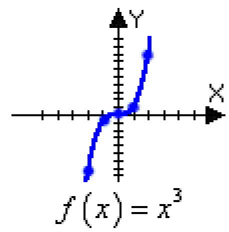
Example 4: Sketch the graph of each function.

(a) $h(x) = \frac{1}{4}x^3$ (b) $g(x) = 3|x|$

Solution (a):

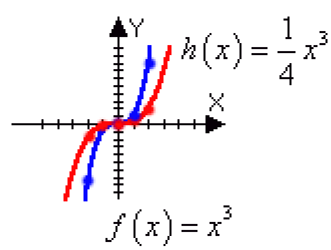
Step 1: Since our function has an x^3 in it, we will use the library function $f(x) = x^3$. Choose a set of points from the graph of $f(x) = x^3$ to work with.

(x, y)
$(-2,-8)$
$(-1,-1)$
$(0,0)$
$(1,1)$
$(2,8)$



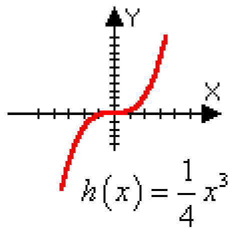
Step 2: The function $h(x) = \frac{1}{4}x^3$ is of the form $y = a\,f(x)$ with $0 < a < 1$, so we know the graph of $h(x)$ will be the same as that of $f(x)$, but shrunken by a factor of $\frac{1}{4}$. Thus, we can obtain points on the graph of $h(x)$ by multiplying the y-coordinate of each point from the graph of $f(x)$ by $\frac{1}{4}$.

$f(x)$	$h(x)$
$(-2,-8)$	$(-2,-2)$
$(-1,-1)$	$\left(-1,-\frac{1}{4}\right)$
$(0,0)$	$(0,0)$
$(1,1)$	$\left(1,\frac{1}{4}\right)$
$(2,8)$	$(2,2)$



Example 4 (Continued):

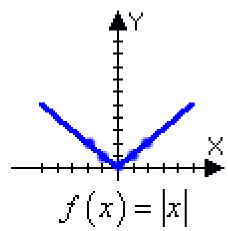
Step 3: Thus we have obtained the graph of $h(x) = \frac{1}{4}x^3$ by transforming the graph of $f(x) = x^3$.



Solution (b):

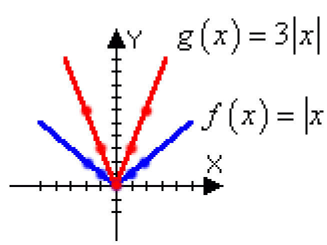
Step 1: Since our function has an $|x|$ in it, we will use the library function $f(x) = |x|$. Choose a set of points from the graph of $f(x) = |x|$ to work with.

(x, y)
$(-2, 2)$
$(-1, 1)$
$(0, 0)$
$(1, 1)$
$(2, 2)$



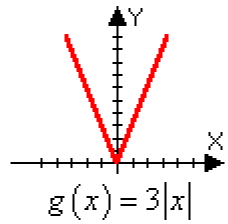
Step 2: The function $g(x) = 3|x|$ is of the form $y = a\,f(x)$ with $a > 1$, so we know the graph of $g(x)$ will be the same as that of $f(x)$, stretched by a factor of 3. Thus, we can obtain points on the graph of $g(x)$ by taking our points from the graph of $f(x) = 3|x|$ and multiplying each of the y-values by 3.

$f(x)$	$g(x)$
$(-2, 2)$	$(-2, 6)$
$(-1, 1)$	$(-1, 3)$
$(0, 0)$	$(0, 0)$
$(1, 1)$	$(1, 3)$
$(2, 2)$	$(2, 6)$



Example 4 (Continued):

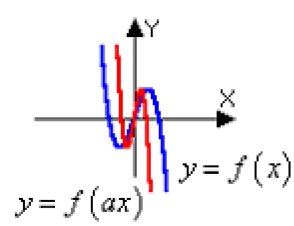
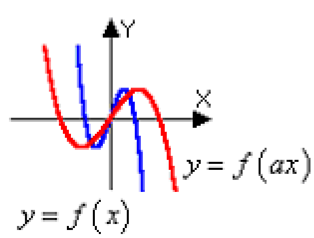
Step 3: Thus we have obtained the graph of $g(x) = 3|x|$ by transforming the graph of $f(x) = |x|$.



Horizontal Stretching and Shrinking:

A horizontal stretching or shrinking is represented in the form $y = f(ax)$. Suppose we know the graph of $y = f(x)$. The value of $f(ax)$ at x is the same as the value of $f(x)$ at ax. Since x is $\frac{1}{a} \cdot ax$, it follows that the graph of $y = f(ax)$ is just the graph of $y = f(x)$ horizontally stretched (or horizontally shrunken) by a factor of $\frac{1}{a}$ if $0 < a < 1$ (if $a > 1$).

Horizontal Stretching and Shrinking of Graphs

Equation	How to Obtain the Graph	What the Graph Looks Like
$y = f(ax)$ $(a > 1)$	Shrink the graph of $y = f(x)$ horizontally by a factor of $\frac{1}{a}$.	
$y = f(ax)$ $(0 < a < 1)$	Stretch the graph of $y = f(x)$ horizontally by a factor of $\frac{1}{a}$.	

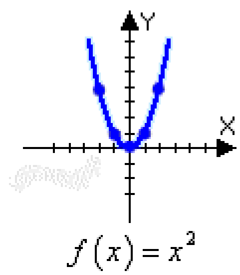
Example 5: Sketch the graph of each function.

(a) $g(x) = \left(\frac{1}{2}x\right)^2$ (b) $h(x) = \sqrt{3x}$

Solution (a):

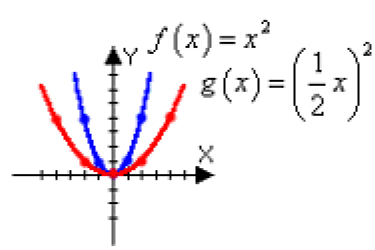
Step 1: Since our function has a $\left(\frac{1}{2}x\right)^2$ in it we will use the library function $f(x) = x^2$. Choose a set of points from the graph of $f(x) = x^2$ to work with.

(x,y)
$(-2,4)$
$(-1,1)$
$(0,0)$
$(1,1)$
$(2,4)$



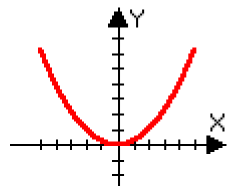
Step 2: The function $g(x) = \left(\frac{1}{2}x\right)^2$ is of the form $y = f(ax)$ with $0 < a < 1$, so we know the graph of $g(x)$ will be the same as that of $f(x)$, but stretched horizontally by a factor of $\frac{1}{a}$, or 2. Thus we can obtain points on the graph of $g(x)$ by multiplying the x-coordinate of each point from the graph of $f(x)$ by 2.

$f(x)$	$g(x)$
$(-2,4)$	$(-4,4)$
$(-1,1)$	$(-2,1)$
$(0,0)$	$(0,0)$
$(1,1)$	$(2,1)$
$(2,4)$	$(4,4)$



Example 5 (Continued):

Step 3: Thus we have obtained the graph of $g(x) = \left(\frac{1}{2}x\right)^2$ by transforming the graph of $f(x) = x^2$.

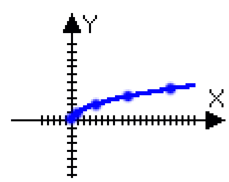


$g(x) = \left(\frac{1}{2}x\right)^2$

Solution (b):

Step 1: Since our function has a $\sqrt{3x}$ in it we will use the library function $f(x) = \sqrt{x}$. Choose a set of points from the graph of $f(x) = \sqrt{x}$ to work with.

$\sqrt{x}$
(0,0)
(1,1)
(4,2)
(9,3)
(16,4)

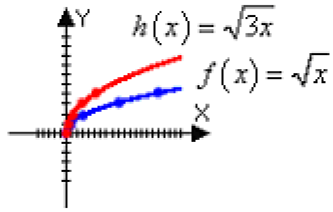


$f(x) = \sqrt{x}$

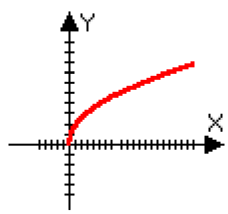
Step 2: The function $h(x) = \sqrt{3x}$ is of the form $y = f(ax)$ with $a > 1$, so we know the graph of $h(x)$ will be the same as that of $f(x)$, but shrunken horizontally by a factor of $\frac{1}{a}$, or $\frac{1}{3}$. Thus we can obtain points on the graph of $h(x)$ by multiplying the x-coordinate of each point from the graph of $f(x)$ by $\frac{1}{3}$.

Example 5 (Continued):

$f(x)$	$h(x)$
$(0,0)$	$(0,0)$
$(1,1)$	$\left(\frac{1}{3},1\right)$
$(4,2)$	$\left(\frac{4}{3},2\right)$
$(9,3)$	$(3,3)$
$(16,4)$	$\left(\frac{16}{3},4\right)$



Step 3: Thus we have obtained the graph of $h(x)=\sqrt{3x}$ by transforming the graph of $f(x)=\sqrt{x}$.



$h(x)=\sqrt{3x}$

Combining Shifting, Stretching, and Reflecting:

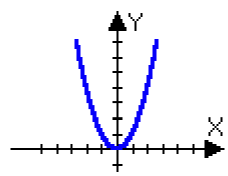
Transformations of functions can be combined to obtain the graphs of more complex functions. For instance, if we know the graph of $y = f(x)$, we can obtain the graph of $y = f(x + 2) - 3$ by shifting the graph of $y = f(x)$ down three units and left two units.

Example 6: Sketch the graph of each function.

(a) $g(x) = \frac{1}{2}x^2 - 3$ (b) $h(x) = -2\sqrt{x+2}$

Solution (a):

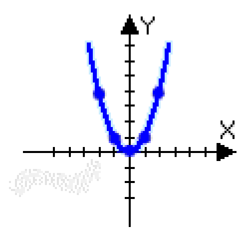
Step 1: Just like in the previous examples, we start by determining which library function best matches our given function. Since our function has an x^2 in it, we will use the library function $f(x) = x^2$.



$f(x) = x^2$

Step 2: Again we choose a set of points from the graph of $f(x) = x^2$ to work with.

(x, y)
$(-2, 4)$
$(-1, 1)$
$(0, 0)$
$(1, 1)$
$(2, 4)$



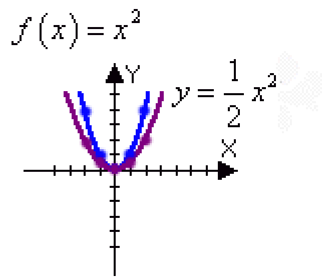
$f(x) = x^2$

Step 3: Now we will perform the transformations one at a time. Our function $g(x) = \frac{1}{2}x^2 - 3$ has two transformations, a shrinking and a vertical shift. Hence, the graph of $g(x)$ will be the same as that of $f(x)$ but shrunken by a factor of $\frac{1}{2}$, and shifted vertically down 3 units.

Example 6 (Continued):

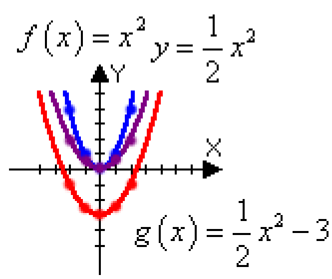
Step 4: First we will perform the shrinking transformation. We want to find the graph of $y = \frac{1}{2}x^2$ using the graph of $f(x) = x^2$. We do this by taking the points from $f(x)$ and multiplying the y-values by $\frac{1}{2}$.

x^2	$\frac{1}{2}x^2$
$(-2,4)$	$(-2,2)$
$(-1,1)$	$\left(-1,\frac{1}{2}\right)$
$(0,0)$	$(0,0)$
$(1,1)$	$\left(1,\frac{1}{2}\right)$
$(2,4)$	$(2,2)$



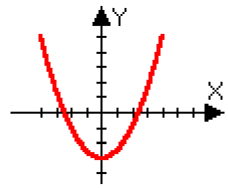
Step 5: Now we will perform the vertical shift transformation using the graph we just found. In other words, we want to find the graph of $g(x) = \frac{1}{2}x^2 - 3$ using the graph of $y = \frac{1}{2}x^2$. We do this by taking the points from the graph and subtracting 3 from each of the *y*-values.

x^2	$\frac{1}{2}x^2$	$\frac{1}{2}x^2 - 3$
$(-2,4)$	$(-2,2)$	$(-2,-1)$
$(-1,1)$	$\left(-1,\frac{1}{2}\right)$	$\left(-1,-\frac{5}{2}\right)$
$(0,0)$	$(0,0)$	$(0,-3)$
$(1,1)$	$\left(1,\frac{1}{2}\right)$	$\left(1,-\frac{5}{2}\right)$
$(2,4)$	$(2,2)$	$(2,-1)$



Example 6 (Continued):

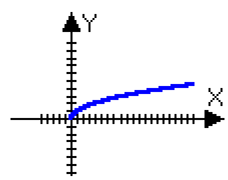
Step 6: Thus we have obtained the graph of $g(x) = \frac{1}{2}x^2 - 3$ by transforming the graph of $f(x) = x^2$.



$g(x) = \frac{1}{2}x^2 - 3$

Solution (b):

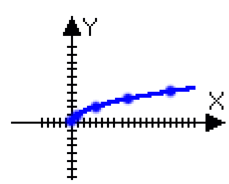
Step 1: Again we start by determining which library function best matches our given function. Since our function has a $\sqrt{x+2}$ in it, we will use the library function $f(x) = \sqrt{x}$.



$f(x) = \sqrt{x}$

Step 2: Again we choose a set of points from the graph of $f(x) = \sqrt{x}$ to work with.

$\sqrt{x}$
$(0,0)$
$(1,1)$
$(4,2)$
$(9,3)$
$(16,4)$



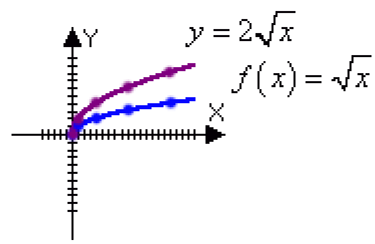
$f(x) = \sqrt{x}$

Example 6 (Continued):

Step 3: Now we will perform the transformations one at a time. This time our function $h(x) = -2\sqrt{x+2}$ has three transformations, a stretch, a reflection and a horizontal shift. Hence, the graph of $h(x)$ will be the same as that of $f(x)$ but stretched by a factor of 2, reflected over the *x*-axis, and shifted horizontally to the left 2 units.

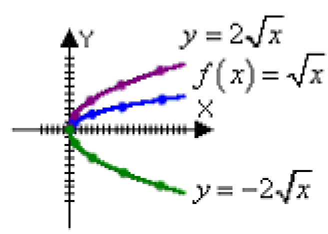
Step 4: First we will perform the stretching transformation. We want to find the graph of $y = 2\sqrt{x}$ using the graph of $f(x) = \sqrt{x}$. We do this by taking the points from $f(x)$ and multiplying the y-values by 2

$\sqrt{x}$	$2\sqrt{x}$
$(0,0)$	$(0,0)$
$(1,1)$	$(1,2)$
$(4,2)$	$(4,4)$
$(9,3)$	$(9,6)$
$(16,4)$	$(16,8)$



Step 5: Next we will perform the reflecting transformation. We will use the graph of $y = 2\sqrt{x}$ to find the graph of $y = -2\sqrt{x}$. To do this, we take each point on the graph of $y = 2\sqrt{x}$ and multiply its *x*-value by -1.

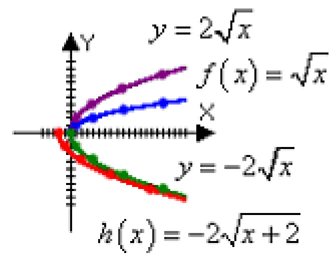
$\sqrt{x}$	$2\sqrt{x}$	$-2\sqrt{x}$
$(0,0)$	$(0,0)$	$(0,0)$
$(1,1)$	$(1,2)$	$(1,-2)$
$(4,2)$	$(4,4)$	$(4,-4)$
$(9,3)$	$(9,6)$	$(9,-6)$
$(16,4)$	$(16,8)$	$(16,-8)$



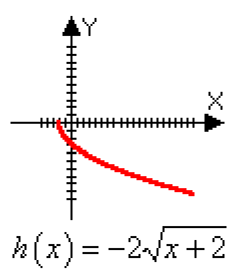
Example 6 (Continued):

Step 6: Finally we will perform the horizontal shift transformation, using the graph we just found. In other words, we want to find the graph of $h(x) = -2\sqrt{x+2}$ using the graph of $y = -2\sqrt{x}$. We do this by taking the points from the graph of $y = -2\sqrt{x}$ and subtracting 2 from each of the *x*-values.

$\sqrt{x}$	$2\sqrt{x}$	$-2\sqrt{x}$	$-2\sqrt{x+2}$
$(0,0)$	$(0,0)$	$(0,0)$	$(-2,0)$
$(1,1)$	$(1,2)$	$(1,-2)$	$(-1,-2)$
$(4,2)$	$(4,4)$	$(4,-4)$	$(2,-4)$
$(9,3)$	$(9,6)$	$(9,-6)$	$(7,-6)$
$(16,4)$	$(16,8)$	$(16,-8)$	$(14,-8)$



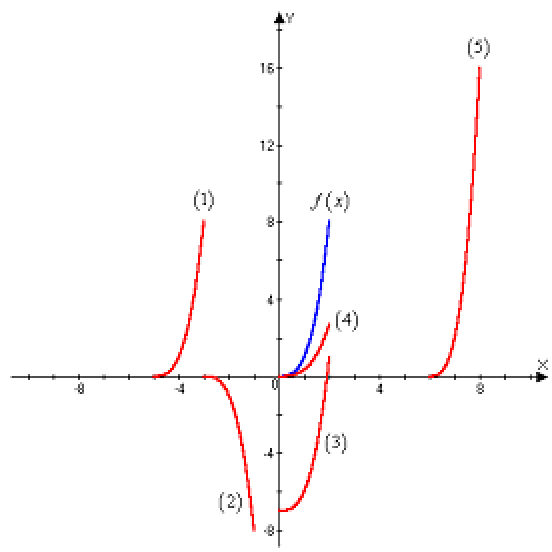
Step 7: Thus we have obtained the graph of $h(x) = -2\sqrt{x+2}$ by transforming the graph of $f(x) = \sqrt{x}$.



Example 7: The graph of $y = f(x)$ is given. Match each equation with its graph.

(a) $y = \frac{1}{3} f(x)$ (b) $y = f(x + 5)$ (c) $y = f(x) - 7$

(d) $y = 2f(x - 6)$ (e) $y = -f(x + 2)$



Solution (a): The graph of $f(x)$ is shown in blue in the above figure. The function $y = \frac{1}{3} f(x)$ is of the form $y = a\, f(x)$ with $0 < a < 1$. Therefore, the graph of $y = \frac{1}{3} f(x)$ will be the same as that of $f(x)$ but shrunken vertically by a factor of $\frac{1}{3}$. Thus, the graph of $y = \frac{1}{3} f(x)$ is (4).

Solution (b): The function $y = f(x + 5)$ is of the form $y = f(x + c)$. Therefore, the graph of $y = f(x + 5)$ will be the same as that of $f(x)$ but shifted to the left 5 units. Thus, the graph of $y = f(x + 5)$ is (1).

Solution (c): The function $y = f(x) - 7$ is of the form $y = f(x) - c$. Therefore, the graph of $y = f(x) - 7$ will be the same as that of $f(x)$ but shifted down 7 units. Thus, the graph of $y = f(x) - 7$ is (3).

Example 7 (Continued):

Solution (d): The function $y = 2\,f\,(x - 6)$ has two forms, $y = a\,f\,(x)$ and $y = f\,(x - c)$. Therefore, the graph of $y = 2\,f\,(x - 6)$ will be the same as that of $f\,(x)$ but stretched by a factor of 2 and then shifted to the right 6 units. Thus, the graph of $y = 2\,f\,(x - 6)$ is (5).

Solution (e): The function $y = -\,f\,(x + 2)$ also has two forms, $y = -\,f\,(x)$ and $y = f\,(x + c)$. Therefore, the graph of $y = -\,f\,(x + 2)$ will be the same as that of $f\,(x)$ but reflected in the x-axis and then shifted to the left 6 units. Thus, the graph of $y = -\,f\,(x + 2)$ is (2).