Inconsistent and Dependent Systems

The elimination method may be used to solve systems of linear equations of more than two variables. The objective is to find the solution of the ordered triple (x, y, z) by using the elimination method covered earlier. The following example demonstrates this idea.

Example 1: **Solve the system of equations.**

$$\begin{aligned} x - y + 4z &= -29 \\ 3x - 3y - z &= -6 \\ 2x - 5y + 6z &= -55 \end{aligned}$$

Solution:

Step 1: Create a 4th equation by eliminating any variable between any two of the given equations. In this case the "x" variable will be eliminated in the combination of equations 1 and 2. this is done by multiplying equation 1 by -3 and then adding it to equation 2.

$$\begin{aligned} 3x - 2y - z &= -6 \\ \underline{-3x + 3y - 12z} &\underline{= 87} \\ y - 13z &= 81 \end{aligned}$$

Step 2: Create a 5th equation by taking the remaining unused original equation and any one of the other two equations and eliminate the same variable as in step 1. In this case equations 1 and 3 are used with the goal of eliminating the "x" variable. Equation 5 will be the sum of equation 3 and -2 times equation 1.

$$\begin{aligned} 2x - 5y + 6z &= -55 \\ \underline{-2x + 2y - 8z} &\underline{= 58} \\ -3y - 2z &= 3 \end{aligned}$$

Example 1 (Continued):

Step 3: Using equations 4 and 5 the process is repeated to eliminate either of the two remaining variables. For this example "y" will be eliminated by first multiplying equation 4 by 3 and then added to equation 5.

$$\begin{array}{r} -3y - 2z = 3 \\ 3y - 39z = 243 \\ \hline -41z = 246 \end{array}$$

Step 4: Solve for z.

$$\begin{aligned} -41z &= 246 \\ \frac{-41z}{-41} &= \frac{246}{-41} \\ z &= -6 \end{aligned}$$

Step 5: Solve for y.

The solution of step 4 is substituted into either equations 4 or 5 to solve for the second variable. In this case equation 5 will be used.

$$\begin{aligned} -3y - 2z &= 3 \\ -3y - 2(-6) &= 3 \\ -3y - (-12) &= 3 \\ -3y + 12 &= 3 \\ -3y + 12 - 12 &= 3 - 12 \\ -3y &= -9 \\ y &= 3 \end{aligned}$$

Example 1 (Continued):

Step 6: Solve for x.

The solutions found in steps 4 and 5 are substituted into any of the original 3 equations to solve for the remaining variable. In this case equation 1 will be used.

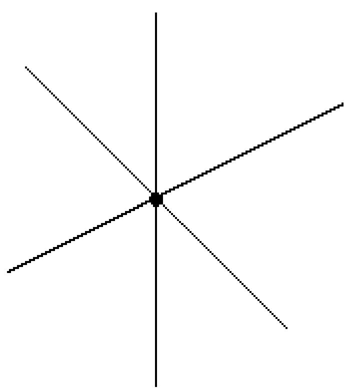
$$
\begin{aligned}
x - y + 4z &= -29 \\
x - (3) + 4(-6) &= -29 \\
x - 3 - 24 &= -29 \\
x - 27 &= -29 \\
x &= -2
\end{aligned}
$$

Step 7: Analysis.

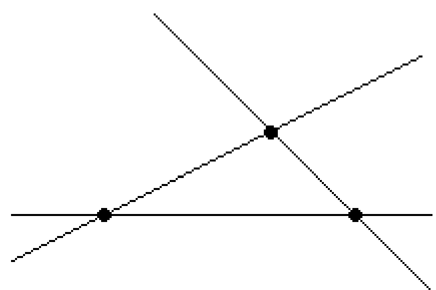
The solutions found for the ordered triple are substituted into the two remaining equations to verify their validity.

$$
\begin{aligned}
3x - 2y - z &= -6 \\
3(-2) - 2(3) - (-6) &= -6 \\
-6 - 6 + 6 &= -6 \\
-6 &= -6
\end{aligned}
\qquad
\begin{aligned}
2x - 5y + 6z &= -55 \\
2(-2) - 5(3) + 6(-6) &= -55 \\
-4 - 15 - 36 &= -55 \\
-55 &= -55
\end{aligned}
$$

The verification is necessary since there are a number of ways in which the system may behave.

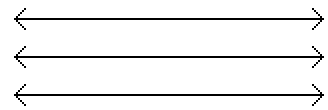


One solution



Multiple solutions

Infinite solutions



No solution

When a system is written only using the coefficients and constants of a system written in a rectangular array, it is known as the matrix form of the system. As with the operations used in the previous example, solutions for the matrix form are found using what are known as elementary row operations. These are listed below with the appropriate notations.

1. *Add a multiple of one row to another.*

   $R_a + kR_b \rightarrow R_a$

2. *Multiply a row by a non zero constant.*

   kR_b

3. *Interchange two rows.*

   $R_a \leftrightarrow R_b$

The goal in using the matrix form is to use variations of these rules to eliminate the first term of the second equation and then the first two terms of the last equation. The following example demonstrates this principle by creating what is known as an echelon form.

Example 2: Solve the system

$$\begin{aligned} x - 2y + z &= 16 \\ 2x - y - z &= 14 \\ 3x + 3y - 4z &= -10 \end{aligned}$$

Example 2 (Continued):

Solution:

Step 1: Write the equation system in matrix form.

$$\begin{bmatrix} 1 & -2 & 1 & 16 \\ 2 & -1 & -1 & 14 \\ 3 & 5 & -4 & -10 \end{bmatrix}$$

Step 2: Eliminate the first term of equations 2 and 3.

$$\begin{bmatrix} 1 & -2 & 1 & 16 \\ 2 & -1 & -1 & 14 \\ 3 & 5 & -4 & -10 \end{bmatrix} \xrightarrow[-3R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & -2 & 1 & 16 \\ 0 & 3 & -3 & -18 \\ 0 & 11 & -7 & -58 \end{bmatrix}$$

Step 3: Eliminate the 2^{nd} term of equation 3.

$$\begin{bmatrix} 1 & -2 & 1 & 16 \\ 0 & 3 & -3 & -18 \\ 0 & 11 & -7 & -58 \end{bmatrix} \quad -11R_2 + R_3 \to R_3 \quad \begin{bmatrix} 1 & -2 & 1 & 16 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

Step 4: Solve for z.

Since the third line of the matrix is equal to $4z = 8$; $z = 2$

Step 5: Solve for y.

Since the second line of the matrix is equal to $y - z = -6$ the value found for z is substituted into the equation to solve for y.

$$\begin{aligned} y - z &= -6 \\ y - (2) &= -6 \\ y &= -4 \end{aligned}$$

Example 2 (Continued):

Step 6: Solve for x.

The solutions for y and z are substituted into the first equation to solve for x.

$$
\begin{aligned}
x-2y+z&=16\\
x-2(-4)+(2)&=16\\
x+8+2&=16\\
x+10&=16\\
x&=6
\end{aligned}
$$

Step 7: Analysis.

The solution for the system is the ordered triple $(6,-4,2)$

In what is termed a reduced echelon form, the matrix of a linear system is manipulated to yield the following format:

$$
\begin{bmatrix}
1 & 0 & 0 & x\\
0 & 1 & 0 & y\\
0 & 0 & 1 & z
\end{bmatrix}
$$

The process used to achieve this format is termed GAUSS-JORDAN ELIMINATION. The following example demonstrates this technique.

Example 3: Solve the system

$$
\begin{aligned}
2x+4y-5z&=37\\
x+3y-4z&=29\\
5x-y+3z&=-20
\end{aligned}
$$

Example 3 (Continued):

Solution:

Step 1: Write the system in the matrix format.

$$\begin{bmatrix} 2 & 4 & -5 & 37 \\ 1 & 3 & -4 & 29 \\ 5 & -1 & 3 & -20 \end{bmatrix}$$

Step 2: Analysis.

Since equation 2 already starts with 1, Equations 1 and 2 positions are exchanged.

$$\begin{bmatrix} 2 & 4 & -5 & 37 \\ 1 & 3 & -4 & 29 \\ 5 & -1 & 3 & -20 \end{bmatrix} \quad R_2 \leftrightarrow R_1 \quad \begin{bmatrix} 1 & 3 & -4 & 29 \\ 2 & 4 & -5 & 37 \\ 5 & -1 & 3 & -20 \end{bmatrix}$$

Step 3: Eliminate the first factor of equations 2 and 3.

$$\begin{bmatrix} 1 & 2 & -4 & 29 \\ 2 & 4 & -5 & 37 \\ 5 & -1 & 3 & -20 \end{bmatrix} \xrightarrow[-5R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 3 & -4 & 29 \\ 0 & -2 & 3 & -21 \\ 0 & -16 & 23 & -165 \end{bmatrix}$$

Step 4: Manipulate the matrix to make the second term of equation 2 equal to one.

$$\begin{bmatrix} 1 & 3 & -4 & 29 \\ 0 & -2 & 3 & -21 \\ 0 & -16 & 23 & -165 \end{bmatrix} \quad -\frac{1}{2}R_2 \to R_2 \quad \begin{bmatrix} 1 & 3 & -4 & 29 \\ 0 & 1 & -\frac{3}{2} & \frac{21}{2} \\ 0 & -16 & 23 & -165 \end{bmatrix}$$

Example 3 (Continued):

Step 5: Manipulate the matrix to make the second term of equations 1 and 3 equal to 0 .

$$\begin{bmatrix} 1 & 3 & -4 & 29 \\ 0 & 1 & -\frac{3}{2} & \frac{21}{2} \\ 0 & -16 & 23 & -165 \end{bmatrix} \xrightarrow[16R_2+R_3\to R_3]{-3R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & -\frac{3}{2} & \frac{21}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Step 6: Manipulate the matrix to make the third term of equation 3 equal to one.

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & -\frac{3}{2} & \frac{21}{2} \\ 0 & 0 & -1 & 3 \end{bmatrix} \quad -1R_3 \to R_3 \quad \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & -\frac{3}{2} & \frac{21}{2} \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Step 7: Manipulate the matrix to make the third term of equations 1 and 3 equal to 0.

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & -\frac{3}{2} & \frac{21}{2} \\ 0 & 0 & 1 & -3 \end{bmatrix} \xrightarrow[\frac{3}{2}R_3+R_2\to R_2]{-\frac{1}{2}R_3+R_1\to R_1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Step 8: Analysis.

The solution ordered triple for this system is seen from the solution matrix of step 7 to be (–1, 6, –3).

The final two examples demonstrate what happens if the system is either inconsistent or dependent.

Example 4: Solve the system

$$\begin{aligned} x-2y+3z&=3 \\ 5x-9y+4z&=2 \\ 2x-4y+6z&=-1 \end{aligned}$$

Solution:

Step 1: Write the system in matrix form.

$$\begin{bmatrix} 1 & -2 & 3 & 3 \\ 5 & -9 & 4 & 2 \\ 2 & -4 & 6 & -1 \end{bmatrix}$$

Step 2: Manipulate the matrix to eliminate the first terms of equations 2 and 3.

$$\begin{bmatrix} 1 & -2 & 3 & 3 \\ 5 & -9 & 4 & 2 \\ 2 & -4 & 6 & -1 \end{bmatrix} \xrightarrow[-2R_1+R_3\to R_3]{-5R_1+R_2\to R_2} \begin{bmatrix} 1 & -2 & 3 & 3 \\ 0 & 1 & -11 & -13 \\ 0 & 0 & 0 & -7 \end{bmatrix}$$

Step 3: Analysis.

Since the third equation = $0x + 0y + 0z = -7$ is a false statement, this system is inconsistent and the has the solution set $\{\varnothing\}$.

Example 5: Solve the system

$$\begin{aligned} x+2y+2z&=9 \\ x-3y-4z&=5 \\ 2x+5y-2z&=14 \end{aligned}$$

Example 5 (Continued):

Solution:

Step 1: Write the system in matrix form.

$$\begin{bmatrix} 1 & 2 & 2 & -9 \\ 1 & -3 & -4 & 5 \\ 2 & 5 & -2 & 14 \end{bmatrix}$$

Step 2: Manipulate the matrix to eliminate the first terms of equations 2 and 3.

$$\begin{bmatrix} 1 & 2 & 2 & -9 \\ 1 & -3 & -4 & 5 \\ 2 & 5 & -2 & 14 \end{bmatrix} \xrightarrow[-2R_1 + R_3 \to R_3]{-1R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 2 & 9 \\ 0 & 1 & -6 & -4 \\ 0 & 1 & -6 & -4 \end{bmatrix}$$

Step 3: Manipulate the matrix to eliminate the second term in equations 1 and 3.

$$\begin{bmatrix} 1 & 2 & 2 & 9 \\ 0 & 1 & -6 & -4 \\ 0 & 1 & -6 & -4 \end{bmatrix} \xrightarrow[-1R_2 + R_3 \to R_3]{-2R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & 14 & 17 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 4: Analysis.

Since the third equation, $0x + 0y + 0z = 0$ and is true for all values, the system is dependant and its solution set consists of infinitely many ordered triples.