Matrix Operations and Their Applications

The dimension of a matrix is defined as a pair of numbers representing the number of rows and columns that a matrix consist of, in the form (R x C). The individual values that a matrix is made of are known as entries and may be specified by their location within the matrix by first identifying the row then column space they occupy.

Using the matrix "S" below:

$$S = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$$

l would be the entry (3, 4) of a 3 x 4 matrix usually denoted as s_{34}.

Matrices are said to be equal if both their dimensions and corresponding values are the same. Matrices may be added or subtracted from each other only if their dimensions are the same. Example 1 demonstrates the addition on matrices while example 2 demonstrates the operation of subtraction.

Example 1: Add the following matrices.

$$a)\ \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \qquad b)\ \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$c)\ \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \qquad d)\ \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

Solution:

Step 1: Analysis.

To add matrices the corresponding terms of each pair of equal matrices are added together. Since the last pair of matrices, (d), are not equal their sum is undefined.

Example 1 (Continued):

Step 2: Add the corresponding values of each matrix pair.

$$a)\ \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+3 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

$$b)\ \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 1+0 & -2+0 \\ 1+0 & 2+0 & 3+0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$c)\ \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+(-1) \\ -3+3 \\ -2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 2: Subtract the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Solution:

Step 1: Analysis.

Since the matrices are equal, their corresponding terms may be subtracted from each other.

Step 2: Subtract the corresponding values of each matrix.

$$\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1-2 & 2-0 & 4-0 \\ -3-1 & 0-(-4) & -1-3 \\ 2-(-1) & 1-3 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 4 \\ -4 & 4 & -4 \\ 3 & -2 & 0 \end{bmatrix}$$

A matrix may be multiplied by a single value. The resulting matrix contains elements known as scalar products. Example 3 demonstrates this concept.

Example 3: **Find the following product:**

$$3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

Solution:

Step 1: Multiply each element of the matrix by 3.

$$3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} (3)(1) & (3)(2) & (3)(4) \\ (3)(-3) & (3)(0) & (3)(-1) \\ (3)(2) & (3)(1) & (3)(2) \end{bmatrix}$$

Step 2: Determine the scalar product.

$$\begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

A matrix may be multiplied by another matrix only if the number of entries in the column of the 1st matrix is equal to the number of entries of the rows of the 2nd matrix. The following is a definition of matrix multiplication.

If $A = \left[a_{ij}\right]$ *is an* $m \times n$ *matrix and* $B = \left[b_{ij}\right]$ *is an* $n \times p$ *matrix,*
then the product AB *is an* $m \times p$ *matrix* $AB = \left[c_{ij}\right]$
where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + ... + a_{in}b_{nj}$

The definition indicates a row-by-column multiplication, where the entry in the i^{th} row j^{th} column of the product AB is obtained by multiplying the entries in the i^{th} row of A by the corresponding j^{th} column of B and then adding the result. Example 4 will demonstrate this principle.

Example 4: Find the product of

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \text{ and } \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

Solution:

Step 1: Analysis.

Since the number of columns of the first matrix is the same number as the rows of the second, the two matrices are able to be multiplied resulting in a 2 x 3 matrix.

Step 2: Perform the row-by-column operations.

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \times \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

$$c_{11} = (1)(-2) + (0)(1) + (3)(-1) = -2 + 0 + (-3) = -5$$
$$c_{12} = (1)(4) + (0)(0) + (3)(1) = 4 + 0 + 3 = 7$$
$$c_{13} = (1)(2) + (0)(0) + (3)(-1) = 2 + 0 + (-3) = -1$$
$$c_{21} = (2)(-2) + (-1)(1) + (-2)(-1) = -4 + (-1) + 2 = -3$$
$$c_{22} = (2)(4) + (-1)(0) + (-2)(1) = 8 + 0 + (-2) = 6$$
$$c_{23} = (2)(2) + (-1)(0) + (-2)(-1) = 4 + 0 + 2 = 6$$

Step 3: Write the products into a matrix form.

$$\begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}$$