Polynomial and Rational Inequalities

This section will explore how to solve inequalities that are either in rational or polynomial form.

Example 1. Solve the equation \( x^2 < x + 6 \) and graph the solution on a number line.

Step 1. Write the equation in standard form.
\[
\begin{align*}
x^2 &< x + 6 \\
x^2 - x - 6 &< x + 6 - x - 6 \\
x^2 - x - 6 &< 0
\end{align*}
\]

Step 2. Find the zeros of the equation (the values of \( x \) that make the equation equal to zero).
\[
\begin{align*}
x^2 - x - 6 &< 0 \\
x^2 - x - 6 &= 0 \\
(x - 3)(x + 2) &= 0 \\
x - 3 &= 0 \text{ or } x + 2 = 0 \\
x &= 3 \text{ or } x = -2
\end{align*}
\]

Step 3. Organize data.

The values found in Step 2, \((3, 0)\) and \((-2, 0)\) are plotted on a number line and the number line is divided into sections (intervals) with boundaries passing through these points:

These values are known as critical points because the boundaries of any intervals pass through them.
Example 1 (Continued):

**Step 4.** Test each interval for its sign values.

\[ (-\infty, -2) ; \quad (-2, 3) \quad \text{and} \quad (3, \infty) \]

To find the sign value of each interval, select any point within the interval (EXCEPT THE CRITICAL POINTS) and substitute the value for \( x \) in the factored form of the polynomial.

**Test point for the interval \((-\infty, -2)\):** \( x = -5 \)

\[
(x - 3)(x + 2) \\
(-5 - 3)(-5 + 2) \\
(-8)(-3) = 24
\]

The product found was positive, therefore the interval is also positive.

**Test point for the interval \((-2, 3)\):** \( x = 0 \)

\[
(x - 3)(x + 2) \\
(0 - 3)(0 + 2) \\
(-3)(2) = -6
\]

The product found was negative, therefore the interval is also negative.

**Test point for the interval \((3, \infty)\):** \( x = 5 \)

\[
(x - 3)(x + 2) \\
(5 - 3)(5 + 2) \\
(2)(7) = 14
\]

The product found was positive, therefore the interval is also positive.
Example 1 (Continued):

Step 5. Graph.

Since the problem desires values that are less than zero they will be negative in nature. Since the interval (-2, 3) contains the only negative results, this interval and all of its values are the only solution. Note that the values of $x = -2$ and $x = 3$ are not part of the solution because these values make the equation equal to zero when they are used, not less than zero. The graph of the solution is:

Example 2. Solve \( \frac{2x-7}{x-5} \leq 3 \) and graph the solution.

Step 1. Write the equation in standard form.

\[
\frac{2x-7}{x-5} \leq 3
\]

\[
\frac{2x-7}{x-5} - 3 \leq 3 - 3
\]

\[
\frac{2x-7}{x-5} - 3 \leq 0
\]

\[
\frac{2x-7}{x-5} - \left( \frac{3}{1} \right) \left( \frac{x-5}{x-5} \right) \leq 0
\]

\[
2x - 7 - (3x-15) \leq 0
\]

\[
2x - 7 - 3x + 15 \leq 0
\]

\[
\frac{-x + 8}{x-5} \leq 0
\]
Example 2 (Continued):

Step 2. Find the critical points.

Note that in addition to the values that make the equation equal to zero, in rational expressions a critical point will involve those values that cannot be used (i.e. the values that would make the denominator zero).

\[
\frac{-x+8}{x-5} \leq 0
\]

\[-x+8 = 0 \quad \text{or} \quad x - 5 = 0
\]

\[8 = x \quad \text{or} \quad x = 5
\]

\[\therefore\] The critical points for this equation are (8, 0) and (5, 0)

Step 3. Organize data.

Since the equation will allow zero to be a solution, the critical point (8, 0) is part of the solution set, whereas the point (5, 0) is not. The reason is the substitution of \(x = 5\) would result in a division by zero. As can be seen on the number line \((-\infty, 5), (5, 8)\) and \([8, \infty)\) are the intervals to be inspected.
Example 2 (Continued):

Step 4. Test the intervals for the sign values.

Test point for the interval \((-\infty, 5]\): \(x = 0\)

\[
\frac{-x + 8}{x - 5} < 0 \\
\frac{-(0) + 8}{0 - 5} < 0 \\
\frac{-8}{-5} < 0
\]

This value is negative so the intervals values are also negative. Therefore this interval is part of the solution.

Test point for the interval \((5, 8]\): \(x = 6\)

\[
\frac{-x + 8}{x - 5} < 0 \\
\frac{-(6) + 8}{6 - 5} < 0 \\
\frac{2}{1} < 0
\]

This value is positive so the intervals values are also positive. Therefore this interval is not part of the solution.

Test point for the interval \([8, \infty)\): \(x = 10\)

\[
\frac{-x + 8}{x - 5} < 0 \\
\frac{-(10) + 8}{10 - 5} < 0 \\
\frac{-2}{5} < 0
\]

This value is negative so the intervals values are also negative. Therefore this interval is part of the solution.
Example 2 (Continued):

Step 5. Graph the solution.

From Step 4 it is determined that the graph and solutions for the equation are:

This may be written in inequality notation as \(-\infty < x < 5\) or \(x \geq 8\) and in interval notation as \((-\infty, 5) \cup [8, \infty)\).