Rational Functions

A rational function is a function of the form

$$r(x) = \frac{P(x)}{Q(x)}$$

where *P* and *Q* are polynomials. We assume that *P*(*x*) and *Q*(*x*) have no factors in common, and *Q*(*x*) is not the zero polynomial.

Rational Functions and Asymptotes:

Remember that a fraction is undefined if there is a zero in the denominator. Rational functions are not defined for those values of *x* for which the denominator is zero. When graphing a rational function, we must pay special attention to the behavior of the graph near those *x*-values. The simplest rational function with a variable denominator is $r(x) = \frac{1}{x}$.

Example 1: Sketch a graph of the rational function $r(x) = \frac{1}{x}$.

Solution:

Step 1: First we note that the function *r* is not defined for *x* = 0. The number 0 cannot be used as a value of *x*, but for graphing it is helpful to find the values of *f* (*x*) for some values of *x* close to 0.

x	$r(x)$
-0.1	-10
-0.01	-100
-0.00001	$-100,000$
↑ approaching 0^-	↑ approaching $-\infty$

x	$r(x)$
0.1	10
0.01	100
0.00001	$100,000$
↑ approaching 0^+	↑ approaching ∞

We describe this behavior in words and in symbols as follows. The first table shows that as *x* approaches 0 from the left, the values of *r* (*x*) decrease without bound. In symbols,

Example 1 (Continued):

Step 1:

$r(x) \to -\infty$ as $x \to 0^-$
"*y* approaches negative infinity as *x* approaches 0 from the left"

The second table shows that as *x* approaches 0 from the right, the values of *r* (*x*) increase without bound. In symbols,

$r(x) \to \infty$ as $x \to 0^+$
"*y* approaches infinity as *x* approaches 0 from the right"

Step 2: Next, we want to look at the end behavior of the graph of $r(x) = \frac{1}{x}$.

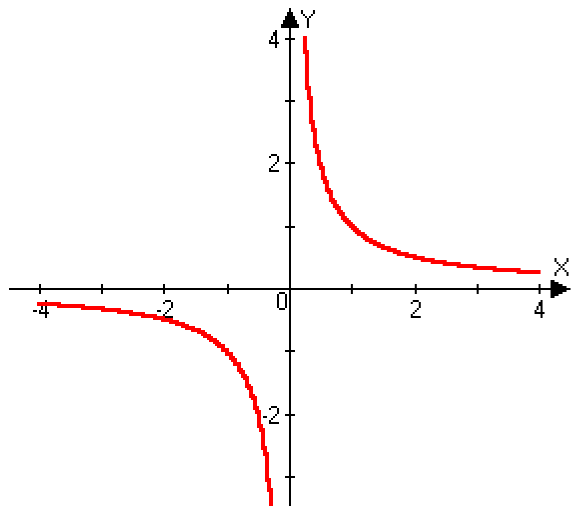
x	$r(x)$
-10	-0.1
-100	-0.01
$-100,000$	-0.00001
↑ approaching $-\infty$	↑ approaching 0

x	$r(x)$
10	0.1
100	0.01
$100,000$	0.00001
↑ approaching ∞	↑ approaching 0

These tables show that as $|x|$ becomes large, the value of *r* (*x*) gets closer and closer to zero.

Example 1 (Continued):

Step 3: Using the information in these tables and plotting a few additional points, we obtain the graph shown below.



$r(x) = \frac{1}{x}$

Unlike a polynomial function, a rational function is usually not continuous. In Example 1, the line $x = 0$ is called a *vertical asymptote* of the graph, and the line $y = 0$ is a *horizontal asymptote*. Another type of asymptote that will be found in later examples is a *slant asymptote* or *oblique asymptote*. It is a strait line that is neither horizontal nor vertical, and has the form $y = ax + b$. Informally speaking, an asymptote of a function is a line that the graph of a function gets closer and closer to as one travels along that line in either direction.

Definition of Asymptotes:

1. The line $x = a$ is a **vertical asymptote** of the function $y = f(x)$ if

$$y \to \infty \text{ or } y \to -\infty \quad \text{as} \quad x \to a^+ \text{ or } x \to a^-$$

2. The line $y = b$ is a **horizontal asymptote** of the function $y = f(x)$ if

$$y \to b \quad \text{as} \quad x \to \infty \text{ or } x \to -\infty$$

3. The line $y = ax + b$ is a **slant asymptote** or **oblique asymptote** of the function $y = f(x)$ if

$$y = ax + b \quad \text{as} \quad x \to \infty \text{ or } x \to -\infty$$

Asymptotes are not part of the graph of a function, but are important aids in graphing the function. Typically, they are drawn as dashed lines to distinguish them from the graph of the function. The following is the procedure for finding asymptotes.

Asymptotes of Rational Functions:

Let r be the rational function

$$r(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x + b_0}$$

1. The vertical asymptotes of r are the lines $x = a$, where a is a zero of the denominator.

2. (a) If $n < m$, then r has horizontal asymptote $y = 0$.

   (b) If $n = m$, then r has horizontal asymptote $y = \frac{a_n}{b_m}$.

   (c) If $n > m$, then r has no horizontal asymptote.

3. If $n = m + 1$, then r has oblique asymptote $y = ax + b$, where $ax + b$ is the quotient obtained when the numerator of the rational function is divided by the denominator, using long division.

Example 2: Find all of the asymptotes of the following rational functions.

(a) $r(x) = \frac{2x}{x+3}$

(b) $s(x) = \frac{1}{x^2+9}$

(c) $t(x) = \frac{x^4}{(x-4)(x+1)}$

(d) $r(x) = \frac{x^3+2x}{x^2}$

Solution (a):

Step 1: We start by finding the vertical asymptotes. These are the lines $x = a$, where a is a zero of the denominator of $r(x) = \frac{2x}{x+3}$. Since $x = -3$ is the only zero of the denominator of r, we have one vertical asymptote at $x = -3$.

Example 2 (Continued):

Step 2: Next we will find the horizontal or oblique asymptote, if it exists. The degree of the numerator of $r(x)=\dfrac{2x}{x+3}$ is n = 1, and the degree of the denominator is m = 1. Since $n = m$, r has a horizontal asymptote at

$$y=\frac{a_n}{b_m}=\frac{2}{1}=2.$$

Solution (b):

Step 1: Vertical Asymptotes: These are the lines $x = a$, where a is a zero of the denominator of $s(x)=\dfrac{1}{x^2+9}$. The zeros of $x^2 + 9$ are $3i$ and $-3i$. Since both of the zeros are imaginary, s has no vertical asymptotes.

Step 2: Horizontal / Oblique Asymptote: The degree of the numerator of $s(x)=\dfrac{1}{x^2+9}$ is n = 0, and the degree of the denominator is m = 2. Since $n < m$, s has a horizontal asymptote at y = 0.

Solution (c):

Step 1: Vertical Asymptotes: The zeros of the denominator of $t(x)=\dfrac{x^4}{(x-4)(x+1)}$ are x = 4 and x = –1, therefore the vertical asymptotes of t are x = 4 and x = –1.

Step 2: Horizontal / Oblique Asymptotes: The degree of the numerator of $t(x)=\dfrac{x^4}{(x-4)(x+1)}$ is n = 4, and the degree of the denominator is m = 2. Since $n > m$, t has no horizontal asymptote, and since $n \neq m + 1$, t has no oblique asymptote.

Example 2 (Continued):

Solution (d):

Step 1: Vertical Asymptotes: The zero of the denominator of $r(x) = \frac{x^3 + 2x}{x^2}$ is $x = 0$, therefore r has one vertical asymptote at $x = 0$.

Step 2: Horizontal / Oblique Asymptotes: The degree of the numerator of $r(x) = \frac{x^3 + 2x}{x^2}$ is $n = 3$, and the degree of the denominator is $m = 2$. Since $n > m$, r has no horizontal asymptote; but $n = m + 1$, and so r has an oblique asymptote at $y = ax + b$, where $ax + b$ is the quotient obtained when the numerator of the rational function is divided by the denominator, using long division.

$$\begin{array}{r|l} & x \\ x^2 & x^3 + 2x \\ & \underline{x^3} \\ & \quad\quad 2x \end{array}$$

Thus, r has an oblique asymptote at $y = x$.

Graphs of Rational Functions:

In general, we use the following guidelines to graph rational functions.

Sketching Graphs of Rational Functions:

1. **Factor.** Factor the numerator and denominator.

2. **Intercepts.** Find the x-intercepts by determining the zeros of the numerator, and the y-intercept from the value of the function at $x = 0$.

3. **Vertical Asymptotes.** Find the vertical asymptotes by determining the zeros of the denominator, and then see if $y \to \infty$ or $y \to -\infty$ on each side of every vertical asymptote.

4. **Horizontal / Oblique Asymptotes.** Find the horizontal or oblique asymptote, if it exists, by looking at the degrees of the numerator and denominator, and following the rules listed above about asymptotes.

5. **Sketch the Graph.** Graph the information provided by the first four steps. Then plot as many additional points as needed to fill in the rest of the graph of the function.

Note that while a graph may never cross a vertical asymptote, it can cross a horizontal asymptote, as long as some part of the graph follows the asymptote

Example 3: Find the intercepts and asymptotes and sketch a graph of the function $s(x) = \dfrac{3x}{x^2 - 16}$.

Solution:

Step 1: Factor: First we factor the numerator and denominator of *s*.

$$s(x) = \frac{3x}{x^2 - 16}$$
$$= \frac{3x}{(x+4)(x-4)}$$

Step 2: Intercepts: Next, we find the *x*-intercepts by determining the zeros of the numerator, and the *y*-intercept from the value of the function at $x = 0$. The numerator $3x$ equals zero at $x = 0$, so the *x*-intercept of *s* is (0, 0). This point is also the *y*-intercept.

Step 3: Vertical Asymptotes: The vertical asymptotes of *s* are $x = -4$, and x = 4. To see if $y \to \infty$ or $y \to -\infty$ on each side of a vertical asymptote, we find the values of $s(x)$ for some values of *x* close to the asymptote, as we did in Example 1. Thus,

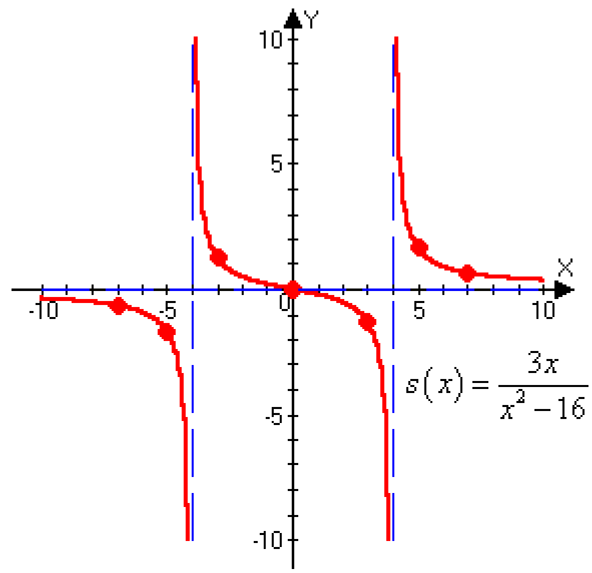
$s(x) \to -\infty$ as $x \to -4^-$ and $s(x) \to \infty$ as $x \to -4^+$
$s(x) \to -\infty$ as $x \to 4^-$ and $s(x) \to \infty$ as $x \to 4^+$

Step 4: Horizontal / Oblique Asymptote: Since the degree of the numerator of *s*, $n = 1$, is less than the degree of the denominator, $m = 2$, *s* has a horizontal asymptote at $y = 0$.

Example 3 (Continued):

Step 5: Sketch the Graph: Graph the information provided by the first four steps. Then plot as many additional points as needed to fill in the rest of the graph of the function.

x	$s(x)$
-7	-0.64
-5	-1.67
-3	1.29
3	-1.29
5	1.67
7	0.64



Example 4: Find the intercepts and asymptotes and sketch a graph of the function $r(x) = \frac{2x^2+3}{x-4}$.

Soultion:

Step 1: Factor: Our function $r(x) = \frac{2x^2+3}{x-4}$ is already in factored form.

Step 2: Intercepts: $2x^2 + 3 = 0$ at $x = i\sqrt{\frac{3}{2}}$ and $x = -i\sqrt{\frac{3}{2}}$, so there is no *x*-intercept of *r*. $r(0) = -\frac{3}{4}$, so the *y*-intercept of *r* is $\left(0, -\frac{3}{4}\right)$.

Step 3: Vertical Asymptotes: The vertical asymptote of *r* is $x = 4$.

$r(x) \to -\infty$ as $x \to 4^-$ and $r(x) \to \infty$ as $x \to 4^+$

Example 4 (Continued):

Step 4: Horizontal / Oblique Asymptote: Since the degree of the numerator of r, $n = 2$, is one more than the degree of the denominator, $m = 1$, r has an oblique asymptote. Using long division, we get

$$\begin{array}{r|l} & 2x + 8 \\ \hline x-4 & 2x^2 + 0x + 3 \\ & \underline{2x^2 - 8x} \\ & \quad 8x + 3 \\ & \quad \underline{8x - 32} \\ & \qquad 35 \end{array}$$

Thus the oblique asymptote is $y = 2x + 8$.

Step 5: Sketch the Graph:

x	$r(x)$
-3	-3
2	-5.5
5	53
10	33.83