Arithmetic Sequences

This section will define and discuss arithmetic sequences, nth term formulas, and the sum formulas for finite arithmetic series.

An arithmetic sequence may be defined as:

> A sequence $\{a_n\}$ is arithmetic if each pair of consecutive terms differs by the same amount, $d = a_i - a_{i-1}$. The number d is called the common difference in the sequence.

(Note: When the formula given in this definition is rewritten as $a_i = a_{i-1} + d$ it is known as a recursive formula because it defines a given term by referencing a proceeding term.)

The following example will show how to find the common difference of an arithmetic sequence.

Example 1: Find the common difference of the arithmetic sequence $a_n = 3 - 5n$.

Solution:

Step 1: Substitute.

Substitute the example formula into the definition formula.

$$\begin{aligned} \boldsymbol{d} &= \boldsymbol{a_i - a_{i-1}} \\ &= \left[\mathbf{3-5}(\boldsymbol{i})\right] - \left[\mathbf{3-5}(\boldsymbol{i}-\mathbf{1})\right] \end{aligned}$$

Step 2: Solve for d.

$$\begin{aligned} \boldsymbol{d} &= \left[\mathbf{3-5}(\boldsymbol{i})\right] - \left[\mathbf{3-5}(\boldsymbol{i}-\mathbf{1})\right] \\ &= \mathbf{3} - \mathbf{5}\boldsymbol{i} - \mathbf{3} + \mathbf{5}\boldsymbol{i} - \mathbf{5} \\ &= -\mathbf{5} \end{aligned}$$

The common difference for this arithmetic sequence is $d = -5$.

The nth term of an arithmetic sequence is defined as:

> The nth term of an arithmetic sequence, whose first term is a_1 and whose common difference is d, is given by the formula $a_n = a_1 + (n - 1)d$.

The following examples will show how to find specific terms of an arithmetic sequence.

Example 2: Find the first four terms and then the twentieth term of the arithmetic sequence whose first term is –1 and whose common difference is 4.

Solution:

Step 1: Substitute.

Since it was given that a_1 = –1 and d = 4, the solutions for the requested terms are found by substitution into the defined formula:

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ a_1 &= -1\,[GIVEN] \\ a_2 &= -1 + (2-1)(4) \\ a_3 &= -1 + (3-1)(4) \\ a_4 &= -1 + (4-1)(4) \\ a_{20} &= -1 + (20-1)(4) \end{aligned}$$

Step 2: Solve.

$$\begin{aligned} a_1 &= -1 \\ a_2 &= -1 + 1(4) = 3 \\ a_3 &= -1 + 2(4) = 7 \\ a_4 &= -1 + 3(4) = 11 \\ a_{20} &= -1 + 19(4) = 75 \end{aligned}$$

Example 3: Find the thirty-eighth term of the arithmetic sequence whose first term is 8 and whose nth term is given by $a_{i+1} = a_i - 7$.

Solution:

Step 1: Solve for d.

$$\begin{aligned} d &= a_{i+1} - a_i \\ &= (a_i - 7) - a_i \\ &= -7 \end{aligned}$$

Example 3 (Continued):

Step 2: Solve.

Since $a_1 = 8$ is given, the thirty-eighth term of the sequence is:

$$\begin{aligned} a_{38} &= a_1 + (n-1)d \\ &= 8 + (38-1)(-7) \\ &= 8 + (37)(-7) \\ &= 8 - 259 \\ &= -251 \end{aligned}$$

Example 4: Find the eighteenth term of the sequence whose seventh-term is 55 and the twenty-second term is 145.

Solution:

Step 1: Solve for d.

The information given is substituted into the nth term formula

$$\begin{aligned} a_7 &= a_1 + 6d \qquad \textit{and} \qquad & a_{22} &= a_1 + 21d \\ 55 &= a_1 + 6d & 145 &= a_1 + 21d \end{aligned}$$

and then d is solved for using systems of linear equations.

$$\begin{array}{r} a_1 + 21d = 145 \\ \underline{-a_1 - 6d = -55} \\ 15d = 90 \\ d = 6 \end{array}$$

Step 2: Solve for a_1.

The value of d is substituted into the first equation to solve for a_1.

$$\begin{aligned} a_1 + 21d &= 145 \\ a_1 + 21(6) &= 145 \\ a_1 + 126 &= 145 \\ a_1 &= 19 \end{aligned}$$

Example 4 (Continued):

Step 3: Solve for a_{18}.

The values found for d and a_1 are used to solve for a_{18}.

$$a_{18} = a_1 + (n-1)d$$
$$a_{18} = 19 + (18-1)(6)$$
$$a_{18} = 19 + 17(6)$$
$$a_{18} = 19 + 102$$
$$a_{18} = 121$$

There are formulas for determining the sum of an arithmetic sequence.

1. $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$
2. $S_n = \frac{n}{2}\left(a_1 + a_n\right)$

The next two examples will show how to use these formulas.

Example 5: Find the sum of the first 99 terms of the arithmetic sequence whose nth term is $a_n = 7 + \frac{n}{2}$.

Solution:

Step 1: Analysis.

The terms given by the problem or that are obvious are:

$$n = 99\,,\ a_1 = 7 + \frac{1}{2}\,,\ and\ a_{99} = 7 + \frac{99}{2}$$

Example 5 (Continued):

Step 2: Substitute and solve.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{99} = \frac{99}{2}\left[\left(7+\frac{1}{2}\right)+\left(7+\frac{99}{2}\right)\right]$$

$$= \frac{99}{2}\left(\frac{15}{2}+\frac{113}{2}\right)$$

$$= \frac{99}{2}\left(\frac{128}{2}\right)$$

$$= 3168$$

Example 6: A small business sells $10,000 worth of products during its first year. The owner of the business has set a goal of increasing annual sales by $7500 each year for 9 years. Assuming that this goal is met, find the total sales during the first 10 years this business is in operation.

Solution:

Step 1: Analysis.

The terms that are given by the problem or that are obvious are:

$$n = 10; a_1 = 10{,}000; \textit{ and } d = 7500$$

Step 2: Substitute and solve.

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$S_{10} = \frac{10}{2}\left[2(10{,}000) + (10-1)(7500)\right]$$

$$= 5\left[20{,}000 + 9(7500)\right]$$

$$= 5[87{,}500]$$

$$= \$437{,}500$$