Binomial Theorem

The formula for combinations is used in determining the binomial coefficient. For nonnegative integers n and r, with n ≥ r, the expression $\binom{n}{r}$ (or sometimes written as $_nC_r$) is called a binomial coefficient and is defined as:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

For example:

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5 \times 4 \times \cancel{3!}}{\cancel{3!} \times 2 \times 1} = \frac{20}{2} = 10$$

The binomial theorem can be used to expand any positive integral power of a binomial in the form of $(a + b)^n$. For any positive integer n the binomial theorem states that:

$$\begin{aligned}(a+b)^n &= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \\ &= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n} b^n\end{aligned}$$

Example 1: Use the binomial formula to expand $(x + 2)^5$.

Solution:

Step 1: Analyze

For $(x + 2)^5$, a = x, b = y, and n = 4

Example 1 (Continued):

Step 2: Substitute.

For $(x + 2)^5$,

$$\left(a+b\right)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$
$$= \sum_{k=0}^{5} \binom{5}{k} \left(x\right)^{5-k} \left(y\right)^k$$

Step 3: Expand.

$$\sum_{k=0}^{5} \binom{5}{k} \left(x\right)^{5-k} \left(2\right)^k$$
$$= \binom{5}{0}\left(x\right)^{5-0}\left(2\right)^0 + \binom{5}{1}\left(x\right)^{5-1}\left(2\right)^1 + \binom{5}{2}\left(x\right)^{5-2}\left(2\right)^2 + \binom{5}{3}\left(x\right)^{5-3}\left(2\right)^3 + \binom{5}{4}\left(x\right)^{5-4}\left(2\right)^4$$
$$+ \binom{5}{5}\left(x\right)^{5-5}\left(2\right)^5$$
$$= \frac{5!\left(x^5\right)\left(1\right)}{0!\left(5-0\right)!} + \frac{5!\left(x^4\right)\left(2^1\right)}{1!\left(5-1\right)!} + \frac{5!\left(x^3\right)\left(2^2\right)}{2!\left(5-2\right)!} + \frac{5!\left(x^2\right)\left(2^3\right)}{3!\left(5-3\right)!} + \frac{5!\left(x^1\right)\left(2^4\right)}{4!\left(5-4\right)!} + \frac{5!\left(x^0\right)\left(2^5\right)}{5!\left(5-5\right)!}$$
$$= \frac{5!\left(x^5\right)}{5!\left(1\right)} + \frac{5\bullet\left(4!\right)\bullet x^4\bullet 2}{1\bullet\left(4!\right)} + \frac{5\bullet 4\bullet\left(3!\right)x^3\bullet 4}{2\bullet 1\bullet 3!} + \frac{5\bullet 4\bullet\left(3!\right)x^2\bullet 8}{3!\bullet 2\bullet 1} + \frac{5\bullet 4!\bullet x\bullet 16}{4!\bullet 1} + \frac{5!\bullet 1\bullet 32}{5!\bullet 1}$$
$$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$$

The binomial theorem can also be used to find just a particular term within the expansion of $(a + b)^n$ by identifying the value of r and then evaluating that term. The value of r will always be equal to the exponent of b, which is one less than the term number. So for example, if you were asked to find the 5th term in a binomial expansion the value of r will be 5 – 1 = 4.

Example 2: Find the sixth term in the expansion of $(3a + 2b)^{12}$.

Solution:

Step 1: Analyze.

For $(3a + 2b)^{12}$, $a = 3a$, $b = 2b$, and $n = 12$. Substituting into the binomial formula yields:

$$\sum_{k=0}^{12}\binom{12}{k}(3a)^{12-k}(2b)^{k}$$

Step 2: Substitute and solve.

For the sixth term $k = 6 – 1 = 5$. Substituting all the appropriate values into the equation formula yields:

$$\begin{aligned}
\sum_{k=0}^{12}\binom{12}{k}(3a)^{12-k}(2b)^{k} &= \sum_{k=0}^{12}\binom{12}{7}(3a)^{12-5}(2b)^{5} \\
&= \frac{12!\bullet 3^7 \bullet 2^5 \bullet a^7 \bullet b^5}{5!(12-5)!} \\
&= \frac{12\bullet 11\bullet 10\bullet 9\bullet 8\bullet 7!\bullet 3^7 \bullet 2^5 \bullet a^7 \bullet b^5}{(7!)\bullet 5\bullet 4\bullet 3\bullet 2\bullet 1} \\
&= \left(\frac{7!}{7!}\right)\left(\frac{12}{4\bullet 3}\right)\left(\frac{10}{5\bullet 2}\right)\left(\frac{11\bullet 9\bullet 8\bullet 2187\bullet 32\bullet a^7\bullet b^5}{1}\right) \\
&= 55,427,328\,a^7b^5
\end{aligned}$$