

# Sequences and Summation Notation

A *sequence* is a set of numbers written in a specific order. For Example:

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

The number  $a_1$  is called the *first term*,  $a_2$  is the *second term*, and  $a_n$  is the *nth term*.

**Definition:** A **sequence** is a function  $f$  whose domain is the set of natural numbers. The values  $f(1), f(2), f(3), \dots$  are called the **terms** of the sequence, and are frequently written as  $a_1, a_2, a_3, \dots$ .

A sequence can be written in a list form, when it is clear what the subsequent terms are, with dots following to indicate the sequence continues indefinitely:

$$5, 10, 15, 20, \dots, a_n, \dots$$

or, it can be written as a formula for the  $n$ th term  $a_n$  of the sequence.

$$a_n = 5n$$

## Finding the Terms of a Sequence:

**Example 1:** Find the first five terms and the 100<sup>th</sup> term of the sequence defined by each formula.

(a)  $a_n = 3n - 6$

(b)  $b_n = n^2$

(c)  $c_n = \frac{n+1}{n+2}$

(d)  $d_n = \frac{(-1)^n}{3^n}$

**Step 1:** To find the first term  $a_1$ , substitute  $n = 1$  into the formula for  $a_n$ .

$$a_1 = 3(1) - 6$$

$$a_1 = -3$$

**Step 2:** To find the second term  $a_2$ , substitute  $n = 2$  into the formula for  $a_n$ .

$$a_2 = 3(2) - 6$$

$$a_2 = 0$$

**Example 1 (Continued):**

**Step 3:** Find the subsequent terms in a similar manner. To find the 100<sup>th</sup> term  $a_{100}$ , substitute  $n = 100$  into the formula for  $a_n$ .

$$a_{100} = 3(100) - 6$$

$$a_{100} = 294$$

**Solution:**

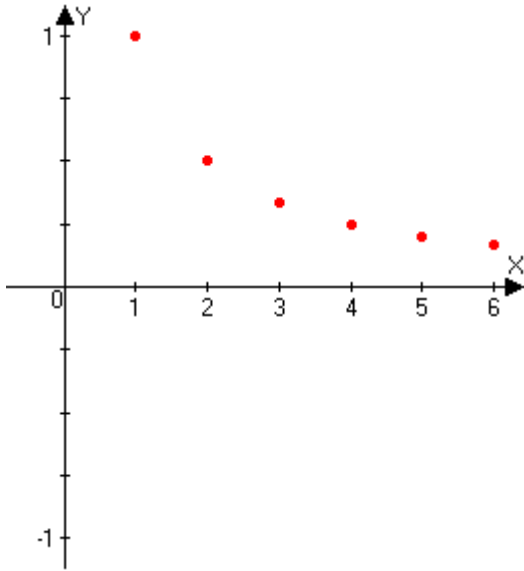
$n$ th term	First five terms	100 <sup>th</sup> term
(a) $3n - 6$	-3, 0, 3, 6, 9	294
(b) $n^2$	1, 4, 9, 16, 25	10000
(c) $\frac{n+1}{n+2}$	$\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$	$\frac{101}{102}$
(d) $\frac{(-1)^n}{3^n}$	$-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, -\frac{1}{243}$	$\frac{1}{3^{100}}$

Note: The  $(-1)^n$  in Example 1(d) causes the successive terms to be alternately negative and positive.

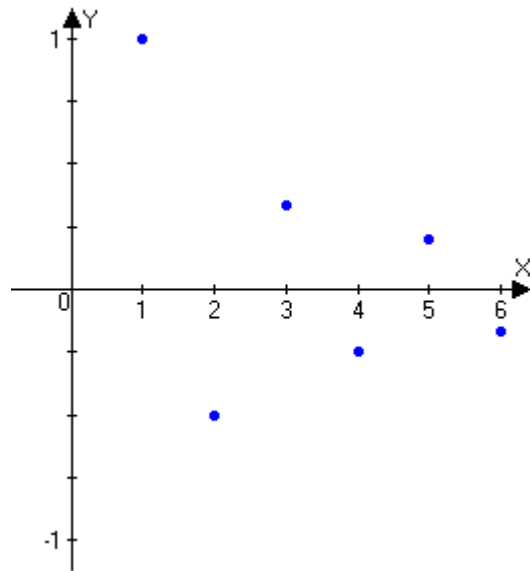
**Graphing a Sequence:**

A sequence is a function whose domain is the natural numbers, so it can be graphed on the Cartesian plane. The graph of a sequence consists of isolated points that are *not* connected. For example, the graph of the sequence  $a_n = \frac{1}{n}$  is shown in Figure 1. Compare this to the graph of

$b_n = \frac{(-1)^{n+1}}{n}$  shown in Figure 2.



**Figure 1:** Graph of  $a_n = \frac{1}{n}$



**Figure 2:** Graph of  $b_n = \frac{(-1)^{n+1}}{n}$

### Finding the Terms of a Recursive Sequence:

A sequence defined in a way such that the  $n$ th term depends on some or all of the preceding terms is called **recursive**.

**Example 2:** Find the first five terms of the sequence defined recursively by

$$a_1 = 0 \text{ and } a_n = 2a_{n-1} + 1$$

**Solution:** Because the sequence is recursive, the  $n$ th term  $a_n$  can be found if the preceding term  $a_{n-1}$  is known. Since we are given the first term  $a_1 = 0$ , we can find the second term, and therefore, then find the third term, and so on.

**Step 1:** We are given the first term  $a_1$ , so now we use it to find the second term  $a_2$ . To do this, substitute  $a_1$  into  $a_n$ .

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2a_1 + 1 \\ &= 2(0) + 1 \\ &= 1 \end{aligned}$$

**Example 2 (Continued):**

**Step 2:** To find the third term  $a_3$ , substitute  $a_2$  into  $a_n$ .

$$\begin{aligned}a_2 &= 1 \\a_3 &= 2a_2 + 1 \\&= 2(1) + 1 \\&= 3\end{aligned}$$

**Step 3:** Find the subsequent terms in the same manner.

$$\begin{aligned}a_4 &= 2a_3 + 1 \\&= 2(3) + 1 \\&= 7 \\a_5 &= 2a_4 + 1 \\&= 2(7) + 1 \\&= 15\end{aligned}$$

Thus, the first five terms of this sequence are:

$$0, 1, 3, 7, 15$$

Note: In order to find the 100<sup>th</sup> term of the recursively defined sequence in Example 2, all 99 preceding terms must first be found.

**The Fibonacci Sequence:**

The **Fibonacci sequence** is the sequence recursively defined by:

$$\begin{aligned}F_1 &= 1, F_2 = 1, \text{ and} \\F_n &= F_{n-1} + F_{n-2}\end{aligned}$$

This sequence is named after a 13<sup>th</sup>-century Italian mathematician who used it to solve a problem about the breeding of rabbits. The Fibonacci sequence also occurs in numerous other applications in nature.

**Example 3:** Find the first 9 terms of the Fibonacci sequence.

**Solution:** Here, the recursive sequence depends on the previous two terms. Therefore, in order to find the  $n$ th term  $F_n$ , the terms  $F_{n-1}$  and  $F_{n-2}$  must be known.

**Step 1:** We know  $F_1$  and  $F_2$ , so we will use them to find the third term  $F_3$ .

$$\begin{aligned}F_1 &= 1 \\F_2 &= 1 \\F_3 &= F_2 + F_1 \\&= 1 + 1 \\&= 2\end{aligned}$$

**Step 2:** Now that we know the third term  $F_3$ , we will use it and the second term  $F_2$  to find the fourth term  $F_4$ .

$$\begin{aligned}F_2 &= 1 \\F_3 &= 2 \\F_4 &= F_3 + F_2 \\&= 2 + 1 \\&= 3\end{aligned}$$

**Step 3:** Find the subsequent terms in the same manner.

$$\begin{aligned}F_5 &= F_4 + F_3 = 3 + 2 = 5 \\F_6 &= F_5 + F_4 = 5 + 3 = 8 \\F_7 &= F_6 + F_5 = 8 + 5 = 13 \\F_8 &= F_7 + F_6 = 13 + 8 = 21 \\F_9 &= F_8 + F_7 = 21 + 13 = 34\end{aligned}$$

Thus, the first 9 terms of the Fibonacci sequence are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34$$

Note: Not all sequences can be defined in the ways discussed above, by a formula or recursively. For example, there is no known formula that produces the sequence of prime numbers:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \dots$$

### Finding the nth Term of a Sequence:

In order to find the nth term of a sequence by looking at its terms, one must try to find a pattern in the given numbers. For instance, consider a sequence that begins

$$2, 4, 8, 16, \dots$$

One pattern for this sequence is  $2^1, 2^2, 2^3, 2^4, \dots$ . Thus, the sequence could be defined by  $a_n = 2^n$ . However, this is not the only sequence whose first four terms are 2, 4, 8, 16. Another sequence might be  $a_n = 2(2^{n-1})$ , or the sequence could be defined recursively as  $a_1 = 2$ ,  $a_n = 2a_{n-1}$ . In other words, the solution is not unique. But, we only need to find the most obvious sequence whose first few terms agree with the given ones.

**Example 4:** Find the nth term of a sequence whose first several terms are given.

(a)  $2, 4, 6, 8, \dots$

(b)  $\frac{2}{3}, \frac{4}{9}, \frac{6}{27}, \frac{8}{81}, \dots$

(c)  $1, -3, 5, -7, \dots$

#### **Solution (a):**

**Step 1:** First, we look for a pattern in the given terms that consists of consecutive integers. One pattern is

$$2(1), 2(2), 2(3), 2(4)$$

**Step 2:** Since these numbers are 2 times the consecutive integers 1, 2, 3, 4, the sequence we are looking for is defined by

$$a_n = 2n$$

Note: This is the definition for the sequence of the even numbers.

**Solution (b):** Since the given terms are fractions, we will look for a pattern in the numerator, and a pattern in the denominator separately.

**Step 1:** First, look for a pattern in the numerators of the fractions. We notice the numerators are the even numbers. Even numbers are of the form  $2n$ , as we learned in example 4(a).

**Example 4 (Continued):**

**Step 2:** Now, look for a pattern in the denominators of the fractions. The numbers in the denominators are 3, 9, 27, 81. A pattern here is

$$3^1, 3^2, 3^3, 3^4$$

These numbers are 3 to the powers 1, 2, 3, 4, so the denominator is of the form  $3^n$ .

**Step 3:** Combining the patterns we found for the numerator and the denominator, we get

$$a_n = \frac{2n}{3^n}$$

**Solution (c):** If the given terms of a sequence are alternating in sign, look at just the numbers for a pattern. Once the pattern is found, multiplying it by either  $(-1)^n$  if the first term is negative, or  $(-1)^{n+1}$  if the first term is positive will create the alternating signs effect.

**Step 1:** Since the given terms are alternating sign, we first look for a pattern in just the numbers, 1, 3, 5, 7. We notice these are the odd numbers. An odd number differs from an even number by one,  $2-1$ ,  $4-1$ ,  $6-1$ ,  $8-1$ .

Thus, odd numbers are of the form  $2n-1$ .

**Step 2:** Now, since the first term is positive, we will multiply  $2n-1$  by  $(-1)^{n+1}$  to define the sequence. Thus, the sequence we are looking for is defined by

$$a_n = (-1)^{n+1} (2n-1)$$

**The Partial Sums of a Sequence:**

A *partial sum* is obtained by adding the terms of a sequence.

Definition: For the sequence  $a_1, a_2, a_3, a_4, \dots, a_n, \dots$

the **partial sums** are

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$\begin{aligned}
S_3 &= a_1 + a_2 + a_3 \\
S_4 &= a_1 + a_2 + a_3 + a_4 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\
&\vdots
\end{aligned}$$

$S_1$  is called the **first partial sum**,  $S_2$  is the **second partial sum**, and so on.  $S_n$  is called the **nth partial sum**. The sequence  $S_1, S_2, S_3, \dots, S_n, \dots$  is called the **sequence of partial sums**.

### Finding the Partial Sums of a Sequence:

**Example 5:** Find the first four partial sums, and the nth partial sum of the sequence defined by

$$a_n = \frac{1}{3^n}.$$

#### Solution:

**Step 1:** To find the first partial sum of the sequence, we will calculate the first term of the sequence.

$$a_1 = \frac{1}{3}$$

The first term of the sequence is the first partial sum of the sequence. Thus

$$S_1 = a_1 = \frac{1}{3}$$

**Step 2:** To find the second partial sum, we calculate the first two terms of the sequence.

$$a_1 = \frac{1}{3}, a_2 = \frac{1}{9}$$

Then we add the two terms together. Thus

$$S_2 = a_1 + a_2 = \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$$

**Step 3:** Find the third and fourth partial sums in the same manner. First calculate the necessary terms of the sequence, and then add them together to obtain the partial sum.



**Example 5 (Continued):**

$$a_1 = \frac{1}{3}, a_2 = \frac{1}{9}, a_3 = \frac{1}{27}, a_4 = \frac{1}{81}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{13}{27}$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} = \frac{40}{81}$$

**Step 4:** Now, to find the  $n$ th partial sum, we will treat our first four partial sums like the first terms of a sequence. Then, we find the  $n$ th partial sum in the same way we would find the  $n$ th term of that sequence. The first four partial sums were

$$\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}$$

The numbers in the denominators, 3, 9, 27, 81, are of the form  $3^n$ , as we found in example 3(b). Notice that the numerator in each partial sum is half of one less than the denominator. That is

$$1 = \frac{1}{2}(3-1)$$

$$4 = \frac{1}{2}(9-1)$$

$$13 = \frac{1}{2}(27-1)$$

$$40 = \frac{1}{2}(81-1)$$

Therefore, the numerator is of the form  $\frac{1}{2}(3^n - 1)$ , and finally

$$S_n = \frac{\frac{1}{2}(3^n - 1)}{3^n} = \frac{1}{2} \left( \frac{3^n - 1}{3^n} \right)$$

## Sigma Notation:

Given a sequence  $a_1, a_2, a_3, a_4, \dots$  the sum of the first  $n$  terms can be written using **summation notation**, or **sigma notation**. This notation derives its name from the Greek letter  $\Sigma$  (capital sigma, corresponding to our S for “sum”). Sigma notation is used as follows:

$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + a_4 + \dots + a_n$$

The left side of this expression is read “the sum of  $a_k$  from  $k=1$  to  $k=n$ .” The letter  $k$  is called the **index of summation**, or the **summation variable**. The idea is to replace  $k$  in the expression  $a_k$  with the integers  $1, 2, 3, \dots, n$ , and then add the resulting terms together.

**Example 6:** Find each sum.

(a)  $\sum_{k=1}^5 k$

(b)  $\sum_{k=1}^5 3$

(c)  $\sum_{j=2}^4 \frac{1}{j}$

(d)  $\sum_{m=4}^7 (-1)^m (m)$

**Solution (a):** We read this expression as “the sum of  $k$  from  $k=1$  to  $k=5$ .”

**Step 1:** First, we calculate the individual terms. We do this by replacing the  $k$  in  $a_k$  with the values  $1, 2, 3, 4, 5$ , to find  $a_1, a_2, a_3, a_4, a_5$ .

$$a_k = k$$

$$a_1 = 1$$

$$a_2 = 2$$

$$a_3 = 3$$

$$a_4 = 4$$

$$a_5 = 5$$

**Example 6 (Continued):**

**Step 2:** Now, we add the terms together to get the sum.

$$\begin{aligned}\sum_{k=1}^5 k &= a_1 + a_2 + a_3 + a_4 + a_5 \\ &= 1 + 2 + 3 + 4 + 5 \\ &= 15\end{aligned}$$

**Solution (b):** We read this expression as “the sum of 3 from  $k=1$  to  $k=5$ .”

**Step 1:** First, we calculate the individual terms. In this problem,  $a_k = 3$ , so every term will be 3. Hence,

$$\begin{aligned}a_k &= 3 \\ a_1 &= 3 \\ a_2 &= 3 \\ a_3 &= 3 \\ a_4 &= 3 \\ a_5 &= 3\end{aligned}$$

**Step 2:** Now, we add the terms together to get the sum.

$$\begin{aligned}\sum_{k=1}^5 3 &= a_1 + a_2 + a_3 + a_4 + a_5 \\ &= 3 + 3 + 3 + 3 + 3 \\ &= 15\end{aligned}$$

**Solution (c):** We read this expression as “the sum of  $\frac{1}{j}$  from  $j=2$  to  $j=4$ .”

**Step 1:** In this problem notice the index of summation will start at 2, instead of 1, and end at 4. So, first we will calculate the terms of  $a_j$  from  $a_2$  to  $a_4$ .

$$\begin{aligned}a_j &= \frac{1}{j} \\ a_2 &= \frac{1}{2} \\ a_3 &= \frac{1}{3} \\ a_4 &= \frac{1}{4}\end{aligned}$$

**Example 6 (Continued):**

**Step 2:** Now, we add the terms together to get the sum.

$$\begin{aligned}\sum_{j=2}^4 \frac{1}{j} &= a_2 + a_3 + a_4 \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\ &= \frac{26}{24} \\ &= \frac{13}{12}\end{aligned}$$

**Solution (d):** We read this expression as “the sum of  $(-1)^m(m)$  from  $m=4$  to  $m=7$ .”

**Step 1:** As in example 1(d), the  $(-1)^m$  in the expression will cause successive terms to alternate in sign. Calculate the terms of  $a_m$  from  $a_4$  to  $a_7$ .

$$\begin{aligned}a_m &= (-1)^m(m) \\ a_4 &= (-1)^4(4) = 4 \\ a_5 &= (-1)^5(5) = -5 \\ a_6 &= (-1)^6(6) = 6 \\ a_7 &= (-1)^7(7) = -7\end{aligned}$$

**Step 2:** Now, add the terms together to get the sum.

$$\begin{aligned}\sum_{m=4}^7 (-1)^m(m) &= a_4 + a_5 + a_6 + a_7 \\ &= 4 + (-5) + 6 + (-7) \\ &= -2\end{aligned}$$

**Example 7:** Write each sum using summation notation.

$$(a) \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$
$$(b) -1 + 2 - 3 + 4 - 5 + 6$$

**Solution (a):**

**Step 1:** First, find a pattern in the given terms that consists of consecutive integers. Since the terms here are fractions, look for a pattern in the numerators and the denominators separately. For this problem, the numerators are all 1. In the denominators, a pattern would be 2, 3, 4, 5, 6. To write the pattern in general form, replace the consecutive integers with  $k$ . Hence,

$$a_k = \frac{1}{k}$$

**Step 2:** Next, find where the index of summation begins and ends. Since the first consecutive integer in the pattern is 2 and the last is 6, the index of summation will begin at 2 and end at 6.

**Step 3:** Finally, write the sum using summation notation.

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \sum_{k=2}^6 \frac{1}{k}$$

**Solution (b):**

**Step 1:** Notice in this problem the terms alternate in sign. To find a pattern, we will look at just the numbers. Here, a pattern that consists of consecutive integers is 1, 2, 3, 4, 5, 6. Thus, the general form is  $k$ . Also, similar to example 4(c), multiplying the general form by either  $(-1)^k$  if the first term is negative, or  $(-1)^{k+1}$  if the first term is positive will create the alternating signs effect. Hence,

$$a_k = (-1)^k k$$

**Step 2:** Since the consecutive integers in the pattern begin at 1 and end at 6, the index of summation will go from 1 to 6. Therefore, the sum written in summation form is

$$\sum_{k=1}^6 (-1)^k k$$

## The Properties of Sums:

The following properties are consequences of properties of the real numbers.

Let  $a_1, a_2, a_3, a_4, \dots$  and  $b_1, b_2, b_3, b_4, \dots$  be sequences. Then for every positive integer  $n$  and any real number  $c$ , the following properties hold.

$$1. \sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$$

$$2. \sum_{k=1}^n (a_k - b_k) = \sum_{k=1}^n a_k - \sum_{k=1}^n b_k$$

$$3. \sum_{k=1}^n ca_k = c \sum_{k=1}^n a_k$$

**Example 8:** Use the properties of sums to write  $\sum_{i=1}^6 (i^2 + i)$  as a sum of two summations.

**Solution:**

**Step 1:** Since the given expression contains a  $+$ , we will use the first property of sums. According to the property, the new summations will have the same index of summation as the original one. Also, the new summations will each contain one term of the original one. Hence,

$$\sum_{i=1}^6 (i^2 + i) = \sum_{i=1}^6 i^2 + \sum_{i=1}^6 i$$