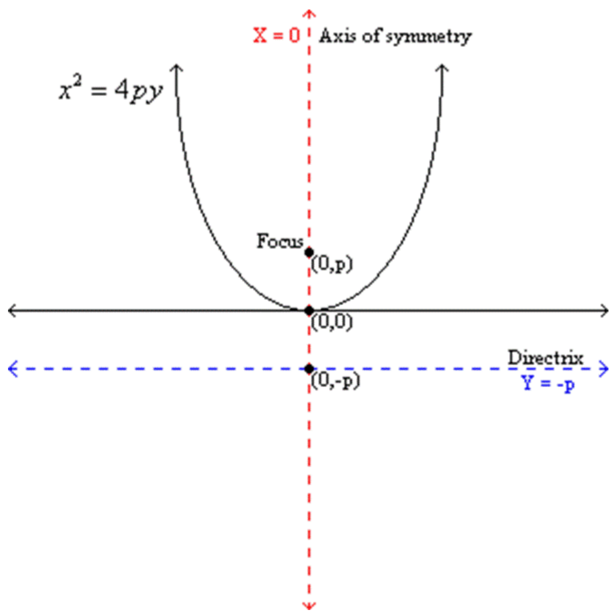
The Parabola and the Circle

The following are several terms and definitions to aid in the understanding of parabolas.

1.) Parabola - A parabola is the set of all points (h, k) that are equidistant from a fixed line called the directrix and a fixed point called the focus (not on the line.)

2.) Axis of symmetry - A line passing through the focus and being perpendicular to the directrix.



3.) The standard equation of a parabola (with the vertex at the origin).

a.) If the *Y* axis is the axis of symmetry.

$x^2 = 4py;\ p \neq 0$

FOCUS: (0, p) DIRECTRIX: $y = -p$

b.) If the *X* axis is the axis of symmetry.

$y^2 = 4px;\ p \neq 0$

FOCUS: (p, 0) DIRECTRIX: $x = -p$

Example 1: Find the focus and directrix and graph the parabola whose equation is $y = -2x^2$.

Solution:

Step 1: Analyze the problem.

Since the quadratic term involves x, the axis is vertical and the standard form $x^2 = 4py$ is used.

Step 2: Apply the formula.

The given equation must be converted into the standard form.

$$y = -2x^2$$

$$\frac{y}{-2} = x^2$$

$$x^2 = -\frac{1}{2}y$$

This means that $4p = -\frac{1}{2}$ or $p = -\frac{1}{8}$.

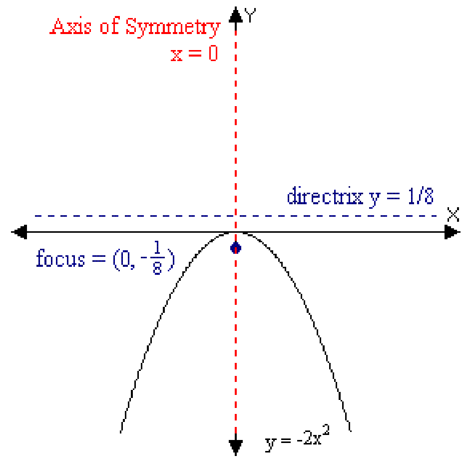
Therefore the focus (0, p) is and the directrix is $\left(0, -\frac{1}{8}\right)$

$$y = -p$$

$$y = -\left(-\frac{1}{8}\right)$$

$$y = \frac{1}{8}$$

Step 3: Graph.



There exists a line parallel to the directrix and passing through the focus called the primary focal chord, |4p|. This chord may be used to help graph the parabola by determining two points on it.

Example 2: Write the standard form of the equation of the parabola with a vertex at the origin and focus at (2, 0). Graph the parabola, including the directrix, the primary focal chord as well as the two points on the graph that they determine.

Solution:

Step 1: Analysis.

Since the vertex is at the origin and the focus is at (2, 0):

1.) The axis is horizontal. Therefore the formula $y^2 = 4px$ is used.
2.) p =2 because the focus, (p, 0) = (2, 0).
3.) The primary focal chord is $|4p| = |(4)(2)| = 8$

Step 2: Find the standard equation.

Since p =2 then:

$y^2 = 4px$
$y^2 = 4(2)x$
$y^2 = 8x$

Step 3: Find the directrix.

The directrix formula is $x = -p$
Since p = 2, then $x = -(2) = -2$
The directrix is $x = -2$.

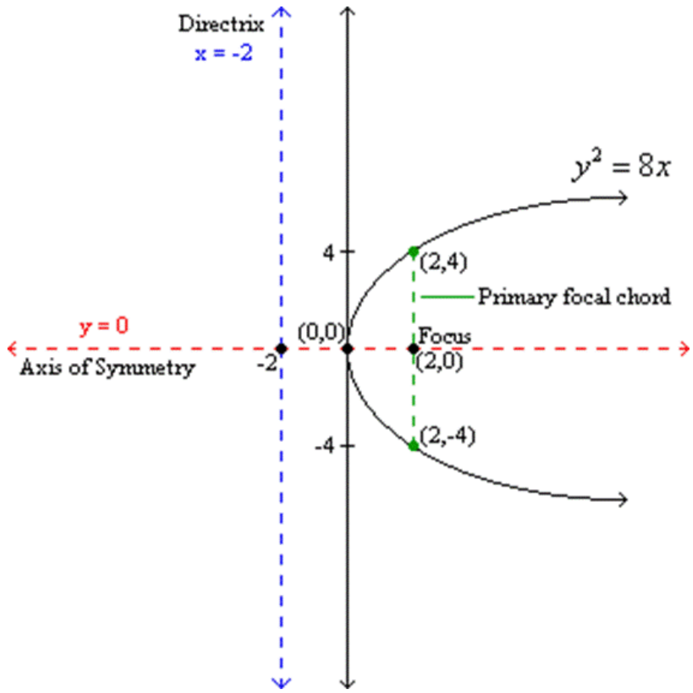
Step 4: Use the primary focal chord to find two points to plot.

The primary focal chord formula is $|4p| = |8| = 8$ units in length.

Therefore the end points of the focal cord are at (2, 4) and (2, -4).

Example 2 (Continued):

Step 5: Graph.



In order to solve problems in which the vertex (h, k) of a parabola is not at the origin, one of the following standard forms should be used, depending on the axis (vertical or horizontal) the original equation indicates.

1.) Symmetry of x = h (vertical): $(x – h)^2 = 4p(y – k)$
The focus will be (h, k + p).
The directrix will be y = k – p.

2.) Symmetry of y = k (horizontal): $(y – k)^2 = 4p(x – h)$
The focus will be (h + p, k).
The directrix will be x = h – p.

Example 3: Find the vertex, focus, directrix, axis of symmetry, primary focal chord and the two points they plot and graph $y^2 + 6y + 8x + 25 = 0$.

Solution:

Step 1: Analysis.

Since the quadratic term involves y the line of symmetry will be y = k. This means that the equation to be used is $(y - k)^2 = 4p(x - h)$.

Step 2: Rewrite the equation into the standard form.

$$y^2 + 6y + 8x + 25 = 0$$
$$y^2 + 6y = -8x - 25$$
$$y^2 + 6y + 9 = -8x - 25 + 9$$
$$(y + 3)^2 = -8x - 16$$
$$(y + 3)^2 = -8(x + 2)$$

Step 3: Sign agreement.

The signs of the solution in step 2 must be made to agree with the signs of the standard equation $(y - k)^2 = 4p(x - h)$.

$$(y + 3)^2 = -8(x + 2)$$
$$(y - (-3))^2 = -8(x - (-2))$$

Step 4: Solutions.

The equation found in step 3, $(y - (-3))^2 = -8(x - (-2))$ indicates that:

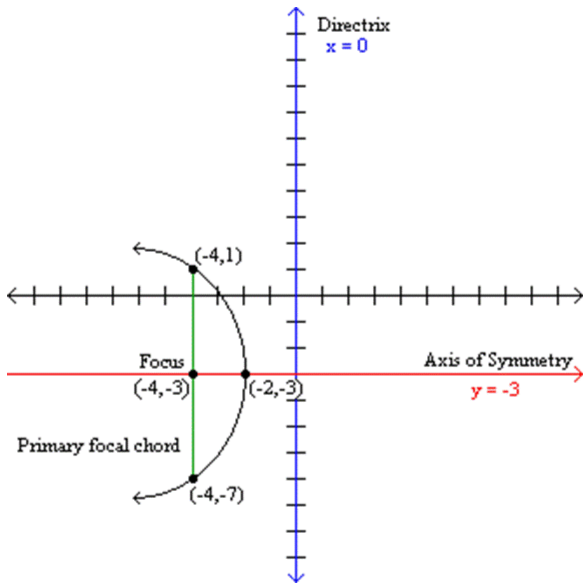
$$k = -3;\ h = -2;\ 4p = -8 \Rightarrow p = -2;\ p < 0$$

This means:

The vertex (h, k) = (-2, -3).
Since $p < 0$, the parabola opens to the left.
The focus (h + p, k) = (-2 + -2, -3) = (-4, -3).
The directrix, x = h – p = -2 – (-2) = 0
Axis of symmetry y = k is y = -3
The primary focal chord |4p| = |-8| = 8
This indicates that (-4, 1) and (-4, -7) are points on the parabola.

Example 3 (Continued):

Step 5: Graph.



The following are several definitions necessary for the understanding of circles.

1.) Circle - A set of points equidistant from a given fixed point on a plane.

2.) Center - The point from which all other points of a circle are equidistant from.

3.) Radius - The distance from the center of a circle to its edge.

4.) Diameter - The distance from one edge of a circle to the other side through the center.

The formula for the radius is $r = \sqrt{(x-h)^2 + (y-k)^2}$, where (h, k) represents the center of the circle and (x, y) a point on the edge.

When both sides of this equation are squared the result is the standard form equation of a circle:

$$r^2 = (x - h)^2 + (y - k)^2$$

Performing the exponentiation and simplifying the equation by getting all of the terms on the same side will give you another form in which the equation of a circle can be expressed. This is called the general form.

$$Ax^2 + By^2 + Cx + Dy + E = 0, \text{ where } A \ \& \ B \neq 0$$

Example 4: Find the equation of a circle, in standard form, having a center at (3, -2) and passing through (-1, 1) on its edge.

Solution:

Step 1: Find the radius.

Substitute the values of the points into the radius formula.

x = -1, y = 1, h = 3, k = -2

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$
$$r = \sqrt{(-1-3)^2 + (1-(-2))^2}$$
$$r = \sqrt{(-4)^2 + (3)^2}$$
$$r = \sqrt{16+9}$$
$$r = \sqrt{25} = 5$$

Step 2: Find the general form of the equation.

Use the radius found in step 1 and the given center point and substitute their values into the standard formula.

r = 5, h = 3, k = -2

$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x - 3)^2 + (y - (-2))^2 = 5^2$$
$$(x - 3)^2 + (y + 2)^2 = 25$$
$$x^2 - 6x + 9 + y^2 + 4y + 4 = 25$$
$$x^2 + y^2 - 6x + 4y + 13 - 25 = 0$$
$$x^2 + y^2 - 6x + 4y - 12 = 0$$

Example 5: Write the given equation of the circle in standard form. Graph.

$$x^2 + y^2 - 2x + 6y + 6 = 0$$

Solution:

Step 1: **Use the technique of completing the square to convert the equation of the circle given into the standard form.**

$$x^2 + y^2 - 2x + 6y + 6 = 0$$
$$(x^2 - 2x) + (y^2 + 6y) = -6$$
$$(x^2 - 2x + 1) + (y^2 + 6y + 9) = -6 + 1 + 9$$
$$(x - 1)^2 + (y + 3)^2 = 4$$

Step 2: **Analyze.**

Recall that the standard form states:

$$(x - h)^2 + (y - k)^2 = r^2$$

This means that the operation involving the y-term should be changed from $(y + 3)^2$ to $(y - (-3))^2$ in order to match the form, $(y - k)^2$. Since the formula expresses the radius *after it's been squared*, (in this case 4), it needs to be expressed as a value squared, 2^2 for this problem.

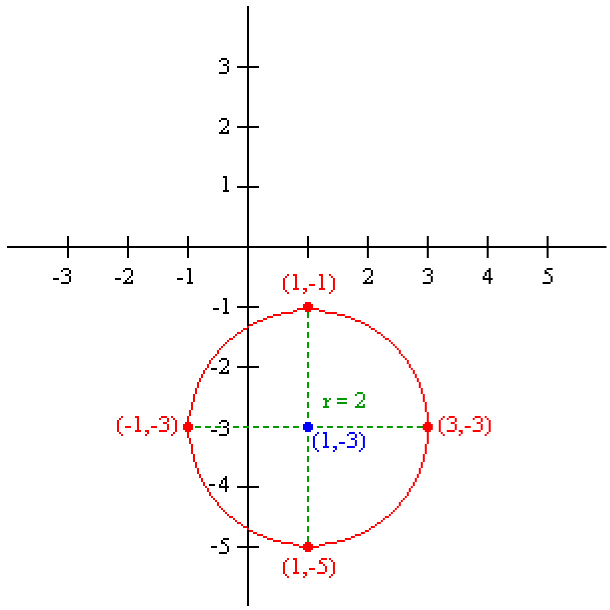
Example 5 (Continued):

Step 3: Graph.

The analysis from step 2 gives the following equation:

$$(x-1)^2 + (y-(-3))^2 = 2^2$$

This means that the center, (h, k) is at (1, -3) and the radius is 2.



Example 6: Find the equation of the circle that has its center at (3, -2) and is tangent to the line 2x + 3y = 13.

Solution:

Step 1: Determine the slope of the tangent line.

$$2x + 3y = 13$$
$$3y = -2x + 13$$
$$y = -\frac{2}{3}x + \frac{13}{3}$$

Therefore, the slope is $-\frac{2}{3}$.

Example 6 (Continued):

Step 2: Find the equation of the radius line.

Since the radius line is perpendicular to the tangent line, its slope is the negative inverse of the slope of the tangent line. Therefore, since the slope of the tangent line is $-\frac{2}{3}$, the slope of the radius line is $\frac{3}{2}$. Since the center point is also on this line, its values and that of the slope of the radius line may be substituted into the slope-intercept formula to solve for the equation of the radius line.

y = mx + b CP = (3, -2) m = $\frac{3}{2}$

$$
\begin{aligned}
&y = mx + b \\
&-2 = \frac{3}{2}(3) + b \\
&-2 = \frac{9}{2} + b \\
&-\frac{4}{2} - \frac{9}{2} = b \\
&-\frac{13}{2} = b
\end{aligned}
$$

Therefore, the equation of the radius line is $y = \frac{3}{2}x - \frac{13}{2}$

Step 3: Rewrite the equation in the standard form.

$$
\begin{aligned}
&y = \frac{3}{2}x - \frac{13}{2} \\
&\frac{3}{2}x - y = \frac{13}{2} \\
&3x - 2y = 13
\end{aligned}
$$

Example 6 (Continued):

Step 4: Solve for the tangent point.

Using systems of equations the tangent point can be solved for.

1.) $2x + 3y = 13$
2.) $3x - 2y = 13$

$$\begin{aligned} 2R_1 &\Rightarrow 4x + 6y = 26 \\ 3R_2 &\Rightarrow \underline{9x - 6y = 39} \\ &\quad\; 13x = 65 \\ &\quad\quad\; x = 5 \end{aligned}$$

Substitute the value of x into one of the two original equations

$2x + 3y = 13$
$2(5) + 3y = 13$
$10 + 3y = 13$
$3y = 3$
$y = 1$

Therefore, the tangent point is (5, 1).

Step 5: Find the radius.

The values of the center point and the tangent point are substituted into the radius formula.

CP = (3, -2) TP = (5, 1)

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$
$$r = \sqrt{(5-3)^2 + (1-(-2))^2}$$
$$r = \sqrt{(2)^2 + (3)^2}$$
$$r = \sqrt{4+9}$$
$$r = \sqrt{13}$$

Example 6 (Continued):

Step 6: Solve for the equation of the circle.

Substitute the radius value, $\left(\sqrt{13}\right)$, and the values of the center point, (3, -2) into the circular formula.

$$(x-h)^2+(y-k)^2=r^2$$
$$(x-3)^2+(y-(-2))^2=\left(\sqrt{13}\right)^2$$
$$(x-3)^2+(y+2)^2=13$$
$$x^2-6x+9+y^2+4y+4=13$$
$$x^2+y^2-6x+4y+9+4-13=0$$
$$x^2+y^2-6x+4y=0$$