Solving Quadratic Equations by Completing The Square

Using the square root property it is possible to solve any quadratic equation written in the form $(x + b)^2 = c$. The key to setting these problems into the correct form is to recognize that $(x + b)^2$ is a perfect square trinomial. To turn the equation given into one that can be solved using the square root property, the following must be done:

Given: $ax^2 + bx + c = 0$

1.) If $a \neq 1$ divide both sides by $a$.
2.) Rewrite the equation so that both terms containing variables are on one side of the equation and the constant is on the other.
3.) Take half of the coefficient of $x$ and square it.
4.) Add the square to both sides.
5.) One side should now be a perfect square trinomial. Write it as the square of a binomial.
6.) Use the square root property to complete the solution.

Example 1. Solve $2a^2 - 4a - 5 = 0$ by completing the square.

Solution

Step 1: Divide the equation by $a$

$$a^2 - 2a - \frac{5}{2} = 0$$

Step 2: Move the constant term to the right side of the equation

$$a^2 - 2a = \frac{5}{2}$$

Step 3: Take half of the coefficient for $x$ and square it

$$\left(\frac{1}{2}\right)\left(\frac{-2}{1}\right) = -1$$

$$(-1)^2 = 1$$

Step 4: Add the square to both sides of the equation

$$a^2 - 2a + 1 = \frac{5}{2} + 1$$
Example 1 (Continued):

Step 5: Factor the perfect square trinomial

\[(a - 1)^2 = \frac{7}{2}\]

Step 6: Take the square root of both sides

\[\sqrt{(a - 1)^2} = \pm \sqrt{\frac{7}{2}} \left( \sqrt{\frac{2}{2}} \right) = \pm \frac{\sqrt{14}}{2}\]

\[(a - 1) = \pm \frac{\sqrt{14}}{2}\]

\[a = 1 \pm \frac{\sqrt{14}}{2}\]

\[a = \frac{2 + \sqrt{14}}{2} \quad \text{or} \quad a = \frac{2 - \sqrt{14}}{2}\]

Example 2. Solve \(9a^2 - 24a = -13\) by completing the square.

Solution

Step 1: Divide the equation by \(a\)

\[a^2 - \frac{24}{9}a = -\frac{13}{9}\]

Step 2: Move the constant term to the right side of the equation

\[a^2 - \frac{8}{3}a = -\frac{13}{9}\]
Example 2 (Continued):

Step 3: Take half of the coefficient for x and square it

\[
\left(\frac{1}{2}\right)\left(-\frac{8}{3}\right) = -\frac{8}{6} = -\frac{4}{3}
\]

\[
\left(-\frac{4}{3}\right)^2 = \frac{16}{9}
\]

Step 4: Add the square to both sides of the equation

\[
a^2 - \frac{8}{3}a + \frac{16}{9} = -\frac{13}{9} + \frac{16}{9}
\]

\[
a^2 - \frac{8}{3}a + \frac{16}{9} = \frac{3}{9}
\]

\[
a^2 - \frac{8}{3}a + \frac{16}{9} = \frac{1}{3}
\]

Step 5: Factor the perfect square trinomial

\[
\left(a - \frac{4}{3}\right)^2 = \frac{1}{3}
\]

Step 6: Take the square root of both sides

\[
\sqrt{\left(a - \frac{4}{3}\right)^2} = \pm \sqrt{\frac{1}{3}} \left(\sqrt{3}\right) = \pm \frac{\sqrt{3}}{3}
\]

\[
a - \frac{4}{3} = \pm \frac{\sqrt{3}}{3}
\]

\[
a = \frac{4}{3} \pm \frac{\sqrt{3}}{3}
\]

\[
a = \frac{4 + \sqrt{3}}{3} \quad \text{or} \quad a = \frac{4 - \sqrt{3}}{3}
\]
Example 3. Solve $9x^2 - 30x + 29$ by completing the square.

Solution

Step 1: Divide the equation by $a$

$$x^2 - \frac{30}{9}x + \frac{29}{9} = 0$$

Step 2: Move the constant term to the right side of the equation

$$x^2 - \frac{10}{3}x = -\frac{29}{9}$$

Step 3: Take half of the coefficient for $x$ and square it

$$\left(\frac{1}{2}\right)\left(-\frac{10}{3}\right) = -\frac{10}{6} = -\frac{5}{3}$$

$$\left(-\frac{5}{3}\right)^2 = \frac{25}{9}$$

Step 4: Add the square to both sides of the equation

$$x^2 - \frac{10}{3}x + \frac{25}{9} = \frac{25}{9} - \frac{29}{9}$$

$$x^2 - \frac{10}{3}x + \frac{25}{9} = -\frac{4}{9}$$

Step 5: Factor the perfect square trinomial

$$\left(x - \frac{5}{3}\right)^2 = -\frac{4}{9}$$
Example 3 (Continued):  

Step 6: Take the square root of both sides 

$$\sqrt{(x - \frac{5}{3})^2} = \pm \sqrt{\frac{-4}{9}}$$

$$x - \frac{5}{3} = \pm \frac{2i}{3}$$

$$x = \frac{5}{3} \pm \frac{2i}{3}$$

$$x = \frac{5 + 2i}{3} \quad \text{or} \quad x = \frac{5 - 2i}{3}$$