Solving Quadratic Equations by Completing The Square

Using the square root property it is possible to solve any quadratic equation written in the form $(x + b)^2 = c$. The key to setting these problems into the correct form is to recognize that $(x + b)^2$ is a perfect square trinomial. To turn the equation given into one that can be solved using the square root property, the following must be done:

Given : $ax^2 + bx + c = 0$

1.) If $a \neq 1$ divide both sides by a.
2.) Rewrite the equation so that both terms containing variables are on one side of the equation and the constant is on the other.
3.) Take half of the coefficient of x and square it.
4.) Add the square to both sides.
5.) One side should now be a perfect square trinomial. Write it as the square of a binomial.
6.) Use the square root property to complete the solution.

Example 1. Solve $2a^2 - 4a - 5 = 0$ by completing the square.

Solution

Step 1: Divide the equation by a

$$a^2 - 2a - \frac{5}{2} = 0$$

Step 2: Move the constant term to the right side of the equation

$$a^2 - 2a = \frac{5}{2}$$

Step 3: Take half of the coefficient for x and square it

$$\left(\frac{1}{2}\right)\left(\frac{-2}{1}\right) = -1$$

$$(-1)^2 = 1$$

Step 4: Add the square to both sides of the equation

$$a^2 - 2a + 1 = \frac{5}{2} + 1$$

Example 1 (Continued):

Step 5: Factor the perfect square trinomial

$$(a-1)^2 = \frac{7}{2}$$

Step 6: Take the square root of both sides

$$\sqrt{(a-1)^2} = \pm\sqrt{\frac{7}{2}}\left(\sqrt{\frac{2}{2}}\right) = \pm\frac{\sqrt{14}}{2}$$

$$(a-1) = \pm\frac{\sqrt{14}}{2}$$

$$a = 1 \pm \frac{\sqrt{14}}{2}$$

$$a = \frac{2}{2} + \frac{\sqrt{14}}{2} \quad \text{or} \quad a = \frac{2}{2} - \frac{\sqrt{14}}{2}$$

$$a = \frac{2+\sqrt{14}}{2} \quad \text{or} \quad a = \frac{2-\sqrt{14}}{2}$$

Example 2. Solve $9a^2 - 24a = -13$ by completing the square.

Solution

Step 1: Divide the equation by a

$$a^2 - \frac{24}{9}a = -\frac{13}{9}$$

Step 2: Move the constant term to the right side of the equation

$$a^2 - \frac{8}{3}a = -\frac{13}{9}$$

Example 2 (Continued):

Step 3: Take half of the coefficient for x and square it

$$\left(\frac{1}{2}\right)\left(\frac{-8}{3}\right) = -\frac{8}{6} = -\frac{4}{3}$$

$$\left(\frac{-4}{3}\right)^2 = \frac{16}{9}$$

Step 4: Add the square to both sides of the equation

$$a^2 - \frac{8}{3}a + \frac{16}{9} = -\frac{13}{9} + \frac{16}{9}$$

$$a^2 - \frac{8}{3}a + \frac{16}{9} = \frac{3}{9}$$

$$a^2 - \frac{8}{3}a + \frac{16}{9} = \frac{1}{3}$$

Step 5: Factor the perfect square trinomial

$$\left(a - \frac{4}{3}\right)^2 = \frac{1}{3}$$

Step 6: Take the square root of both sides

$$\sqrt{\left(a - \frac{4}{3}\right)^2} = \pm\sqrt{\frac{1}{3}}\left(\sqrt{\frac{3}{3}}\right) = \pm\frac{\sqrt{3}}{3}$$

$$a - \frac{4}{3} = \pm\frac{\sqrt{3}}{3}$$

$$a = \frac{4}{3} \pm \frac{\sqrt{3}}{3}$$

$a = \dfrac{4+\sqrt{3}}{3}$ or $a = \dfrac{4-\sqrt{3}}{3}$

Example 3. Solve $9x^2 – 30x + 29$ by completing the square.

Solution

Step 1: Divide the equation by a

$$x^2 - \frac{30}{9}x + \frac{29}{9} = 0$$

Step 2: Move the constant term to the right side of the equation

$$x^2 - \frac{10}{3}x = -\frac{29}{9}$$

Step 3: Take half of the coefficient for x and square it

$$\left(\frac{1}{2}\right)\left(-\frac{10}{3}\right) = -\frac{10}{6} = -\frac{5}{3}$$

$$\left(-\frac{5}{3}\right)^2 = \frac{25}{9}$$

Step 4: Add the square to both sides of the equation

$$x^2 - \frac{10}{3}x + \frac{25}{9} = \frac{25}{9} - \frac{29}{9}$$

$$x^2 - \frac{10}{3}x + \frac{25}{9} = -\frac{4}{9}$$

Step 5: Factor the perfect square trinomial

$$\left(x - \frac{5}{3}\right)^2 = -\frac{4}{9}$$

Example 3 (Continued):

Step 6: Take the square root of both sides

$$\sqrt{\left(x-\frac{5}{3}\right)^2} = \pm\sqrt{\frac{-4}{9}}$$

$$x-\frac{5}{3} = \pm\frac{2i}{3}$$

$$x = \frac{5}{3} \pm \frac{2i}{3}$$

$$x = \frac{5+2i}{3} \qquad \text{or} \qquad x = \frac{5-2i}{3}$$