Factoring expressions and Solving Equations That Are Quadratic in Form

You have already learned how to factor quadratic expressions in the form of $ax^2 + bx + c$ where $a \neq 0$. However, there are instances where you will be faced with trinomials of a degree higher than two but still fit the quadratic form. For example, the trinomial $ax^4 + bx^2 + c$ has a degree power of four but it still fits the quadratic form.

$$ax^4 + bx^2 + c = a(x^2)^2 + b(x^2) + c$$

This substitution pattern can also be applied to situations where there is an algebraic expression in place of $x$ in the quadratic form. Take for example the expression $2(x – 3)^2 – 5(x – 3) – 12$. In this trinomial instead of have $x$ and $x^2$ we have $(x – 3)$ and $(x – 3)^2$.

$$2(x – 3)^2 – 5(x – 3) – 12 = 2u^2 – 5u – 12$$

The following steps can be used to solve equations that are quadratic in form:

1. Let $u$ equal a function of the original variable (normally the middle term)
2. Substitute $u$ into the original equation so that it is in the form $au^2 + bu + c = 0$
3. Factor the quadratic equation using the methods learned earlier
4. Solve the equation for $u$
5. Replace $u$ with the expression of the original variable
6. Solve the resulting equation for the original variable
7. Check for any extraneous solutions

Example 1:  Solve the equation $x^4 – 13x^2 + 36 = 0$.

Solution

Step 1: Let $u$ equal a function of the original variable

In this problem, we would let $u$ equal $x^2$

Step 2: Substitute $u$ into the original equation for the variable expression

Before performing the substitution rewrite $x^4$ as a multiply of $x^2$ which will be replaced by $u$. $x^4 = (x^2)^2$

$$x^4 – 13x^2 + 36 = 0$$

$$(x^2)^2 – 13x^2 + 36 = 0$$

$u^2 – 13u + 36 = 0$$
Example 1 (Continued):

Step 3: Factor the quadratic equation

\[ u^2 - 13u + 36 = 0 \]
\[ (u - 4)(u - 9) = 0 \]

Step 4: Solve the equation for \( u \)

\[ (u - 4)(u - 9) = 0 \]
\[ u - 4 = 0 \quad \text{or} \quad u - 9 = 0 \]
\[ u = 4 \quad \text{or} \quad u = 9 \]

Step 5: Replace \( u \) with the expression of the original variable

\[ u = 4 \quad \text{or} \quad u = 9 \]
\[ x^2 = 4 \quad \text{or} \quad x^2 = 9 \]

Step 6: Solve for the original variable

\[ x^2 = 4 \quad \text{or} \quad x^2 = 9 \]
\[ x^2 - 4 = 0 \quad \text{or} \quad x^2 - 9 = 0 \]
\[ (x - 2)(x + 2) = 0 \quad \text{or} \quad (x - 3)(x + 3) = 0 \]
\[ x - 2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 3 = 0 \quad \text{or} \quad x + 3 = 0 \]
\[ x = 2 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 3 \quad \text{or} \quad x = -3 \]

Step 7: Check for any extraneous solutions

\[ x = 2 \]
\[ x^4 - 13x^2 + 36 = 0 \]
\[ (2)^4 - 13(2)^2 + 36 = 0 \]
\[ 16 - 52 + 36 = 0 \]
\[ 52 - 52 = 0 \]
\[ 0 = 0 \]

\[ x = -2 \]
\[ x^4 - 13x^2 + 36 = 0 \]
\[ (-2)^4 - 13(-2)^2 + 36 = 0 \]
\[ 16 - 52 + 36 = 0 \]
\[ 52 - 52 = 0 \]
\[ 0 = 0 \]
Example 1 (Continued):

\[ x = 3 \]

\[ x^4 - 13x^2 + 36 = 0 \]
\[ (3)^4 - 13(3)^2 + 36 = 0 \]
\[ 81 - 117 + 36 = 0 \]
\[ 117 - 117 = 0 \]
\[ 0 = 0 \]

\[ x = -3 \]

\[ x^4 - 13x^2 + 36 = 0 \]
\[ (-3)^4 - 13(-3)^2 + 36 = 0 \]
\[ 81 - 117 + 36 = 0 \]
\[ 117 - 117 = 0 \]
\[ 0 = 0 \]

Example 2: Solve the equation \( 2x^{2/3} - 7x^{1/3} + 6 = 0 \).

Solution

Step 1: Let \( u \) equal a function of the original variable

In this problem, we would let \( u \) equal \( x^{1/3} \)

Step 2: Substitute \( u \) into the original equation for the variable expression

Before performing the substitution rewrite \( x^{2/3} \) as a multiply of \( x^{1/3} \) which will be replaced by \( u \).

\[ x^{2/3} = (x^{1/3})^2 \]

\[ 2x^{2/3} - 7x^{1/3} + 6 = 0 \]
\[ 2(x^{1/3})^2 - 7x^{1/3} + 6 = 0 \]
\[ 2u^2 - 7u + 6 = 0 \]

Step 3: Factor the quadratic equation

\[ 2u^2 - 7u + 6 = 0 \]
\[ (2u - 3)(u - 2) = 0 \]

Step 4: Solve the equation for \( u \)

\[ (2u - 3)(u - 2) = 0 \]
\[ 2u - 3 = 0 \quad \text{or} \quad u - 2 = 0 \]
\[ 2u = 3 \quad \text{or} \quad u = 2 \]
u = 3/2

Example 1 (Continued):

Step 5: Replace u with the expression of the original variable

\[
\begin{align*}
u &= 3/2 \quad \text{or} \quad u = 2 \\
x^{1/3} &= 3/2 \quad \text{or} \quad x^{1/3} = 2
\end{align*}
\]

Step 6: Solve for the original variable

\[
\begin{align*}
x^{1/3} &= 3/2 \quad \text{or} \quad x^{1/3} = 2 \\
(x^{1/3})^3 &= (3/2)^3 \quad \text{or} \quad (x^{1/3})^3 = (2)^3 \\
x &= 27/8 \quad \text{or} \quad x = 8
\end{align*}
\]

Step 7: Check for any extraneous solutions

\[
x = 27/8
\]

\[
2x^{2/3} - 7x^{1/3} + 6 = 0 \\
2(27/8)^{2/3} - 7(27/8)^{1/3} + 6 = 0 \\
2[(27/8)^{1/3}]^2 - 7(27/8)^{1/3} + 6 = 0 \\
2(3/2)^2 - 7(3/2) + 6 = 0 \\
2(9/4) - 21/2 + 6 = 0 \\
9/2 - 21/2 + 6 = 0 \\
-12/2 + 6 = 0 \\
-6 + 6 = 0 \\
0 = 0
\]

\[
x = 8
\]

\[
2x^{2/3} - 7x^{1/3} + 6 = 0 \\
2(8)^{2/3} - 7(8)^{1/3} + 6 = 0 \\
2[(8)^{1/3}]^2 - 7(8)^{1/3} + 6 = 0 \\
2(2)^2 - 7(2) + 6 = 0 \\
2(4) - 14 + 6 = 0 \\
8 - 14 + 6 = 0 \\
14 - 14 = 0 \\
0 = 0
\]