Exponential Functions with Base $e$

Any positive number can be used as the base for an exponential function, but some bases are more useful than others. For instance, in computer science applications, the base 2 is convenient. The most important base though is the number denoted by the letter $e$.

The number $e$ is irrational, so we cannot write its exact value; the approximate value to 20 decimal places is

$$e \approx 2.71828182845904523536$$

It may seem at first that a base such as 10 is easier to work with, but in certain applications, such as compound interest or population growth, the number $e$ is the best possible base.

The Natural Exponential Function:

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base $e$. It is often referred to as *the* exponential function.

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $y = 2^x$ and $y = 3^x$, as shown below.
Example 1: Graph the function \( y = -e^{x-1} \), not by plotting points, but by starting from the graph of \( y = e^x \) in the above figure. State the domain, range, and asymptote.

Solution:

Step 1: We will use transformation techniques to obtain the graph of 
\( y = -e^{x-1} \). Start with the graph of \( y = e^x \), reflect it in the \( x \)-axis and shift it rightward 1 unit.

Step 2: Since our transformation does not involve a vertical shift of the graph, the horizontal asymptote of \( y = -e^{x-1} \) is the same as that of \( y = e^x \); that is, the horizontal asymptote is the \( x \)-axis, \( y = 0 \).

Looking at the graph, we see that the domain of \( y = -e^{x-1} \) is all real numbers \((-\infty, \infty)\), and the range is \((-\infty, 0)\).

Continuously compounded interest is calculated by the formula

\[
A(t) = Pe^{rt}
\]

where \( A(t) \) = amount after \( t \) years
\( P \) = principal
\( r \) = interest rate
\( t \) = number of years
Example 2: Find the amount after 7 years if $100 is invested at an interest rate of 13% per year, compounded continuously.

Solution:

Step 1: This problem requires that we find an amount that is compounded continuously, thus we will use the continuously compounded interest formula:

\[ A = Pe^{rt} \]

Step 2: The initial amount invested is $100, so \( P = 100 \). The interest rate is 13% per year, so \( r = 0.13 \). The amount will be invested for 7 years, so \( t = 7 \).

Step 3: Now we will substitute the values \( P = 100 \), \( r = 0.13 \), and \( t = 7 \) into the formula for continuously compounded interest.

\[ A(7) = 100e^{0.13(7)} = 248.4322 \]

Step 4: Thus, the amount after 7 years will be $248.43.

Exponential Models of Population Growth:

Population growth is another application of the exponential function. A population experiencing exponential growth increases according to the model

\[ n(t) = n_0e^{rt} \]

where \( n(t) \) = population at time \( t \)
\( n_0 \) = initial size of the population
\( r \) = relative rate of growth (expressed as a proportion of the population)
\( t \) = time

Notice that the formula for population growth is the same as that for continuously compounded interest. In fact, the same principal is at work in both cases: The growth of a population (or an investment) per time period is proportional to the size of the population (or the amount of the investment).
Example 3: The number of bacteria in a culture is given by the function

\[ n(t) = 10e^{0.22t} \]

(a) What is the relative rate of growth of this bacterium population? Express your answer as a percentage.
(b) What is the initial population of the culture (at \( t = 0 \))?
(c) How many bacteria will be in the culture at time \( t = 15 \)?

Solution (a):

Population is modeled using the exponential growth model:

\[ n(t) = n_0e^{rt} \]

where \( r \) is the relative rate of growth. By inspecting the equation we are given, we see that \( r = 0.22 \), or 22%.

Solution (b):

To find the initial population, we find the population at time \( t = 0 \). We do this by substituting \( t = 0 \) into the equation for \( n(t) \).

\[
\begin{align*}
n(t) &= 10e^{0.22t} \\
n(0) &= 10e^{0.22(0)} \\
n(0) &= 10(1) \\
n(0) &= 10
\end{align*}
\]

Thus, the initial population is 10 bacteria.

Note: We could have solved this problem by inspecting the given equation and noticing that \( n_0 \), the initial population size, is 10.

Solution (c):

The number of bacteria at time 15 can be found by substituting \( t = 15 \) into the equation for \( n(t) \).

\[
\begin{align*}
n(t) &= 10e^{0.22t} \\
n(15) &= 10e^{0.22(15)} \\
n(15) &= 10e^{33} \\
n(15) &\approx 10(27.1126) \\
n(15) &\approx 271.1264
\end{align*}
\]

Thus, the population at time 15 is 271 bacteria.

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Example 4: The Ewok population on the planet Endor has a relative growth rate of 3% per year, and it is estimated that the population is 6,500.

(a) Find a function that models the population \(t\) years from now.
(b) Use the function from part (a) to estimate the Ewok population in 8 years.
(c) Sketch the graph of the population function.

Solution (a):

Step 1: To find a function that models the Ewok population, we will use the exponential growth model

\[ n(t) = n_0e^{rt} \]

To use the model, we will need to determine what the values \(n_0\) and \(r\).

Step 2: Since we are not explicitly told when our time starts, we can assume the population was estimated to be 6,500 at time \(t = 0\). Thus our initial population is \(n_0 = 6500\).

Step 3: We are told in the problem that the relative growth rate is 3% per year, so \(r = 0.03\).

Step 4: Now we will substitute the values \(n_0 = 6500\) and \(r = 0.03\) into the formula for the exponential growth model to find the function that models the population \(t\) years from now.

\[ n(t) = n_0e^{rt} \]
\[ n(t) = 6500e^{0.03t} \]

Solution (b):

An estimate of the Ewok population in 8 years can be found by substituting \(t = 8\) into the equation for \(n(t)\).

\[ n(t) = 6500e^{0.03t} \]
\[ n(8) = 6500e^{0.03(8)} \]
\[ n(8) = 6500e^{0.24} \]
\[ n(8) \approx 6500(1.2710) \]
\[ n(8) \approx 8263.1195 \]

Thus, the population in 8 years will be 8263.
Example 4 (Continued):

Solution (c):

**Step 1:** We will graph the population function $n(t) = 6500e^{0.03t}$ by first making a table of values.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$n(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>4815.32</td>
</tr>
<tr>
<td>0</td>
<td>6500.00</td>
</tr>
<tr>
<td>5</td>
<td>7551.92</td>
</tr>
<tr>
<td>8</td>
<td>8283.12</td>
</tr>
<tr>
<td>20</td>
<td>11843.77</td>
</tr>
<tr>
<td>40</td>
<td>21580.76</td>
</tr>
</tbody>
</table>

**Step 2:** Now we will plot the points found in the previous step, and draw a smooth curve connecting them.