Exponential Functions with Base *e*

Any positive number can be used as the base for an exponential function, but some bases are more useful than others. For instance, in computer science applications, the base 2 is convenient. The most important base though is the number denoted by the letter *e*.

The number *e* is irrational, so we cannot write its exact value; the approximate value to 20 decimal places is

$$e \approx 2.71828182845904523536$$

It may seem at first that a base such as 10 is easier to work with, but in certain applications, such as compound interest or population growth, the number *e* is the best possible base.

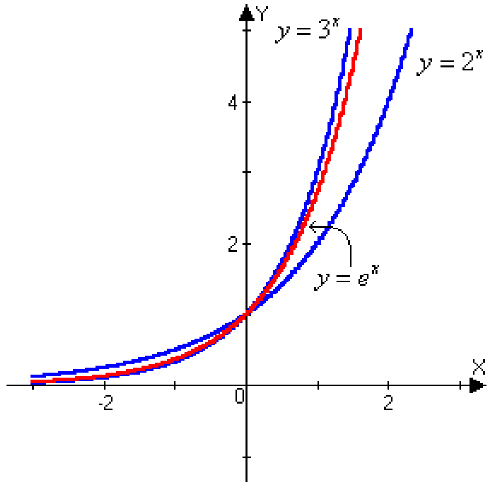
The Natural Exponential Function:

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base *e*. It is often referred to as *the* exponential function.

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $y = 2^x$ and $y = 3^x$, as shown below.



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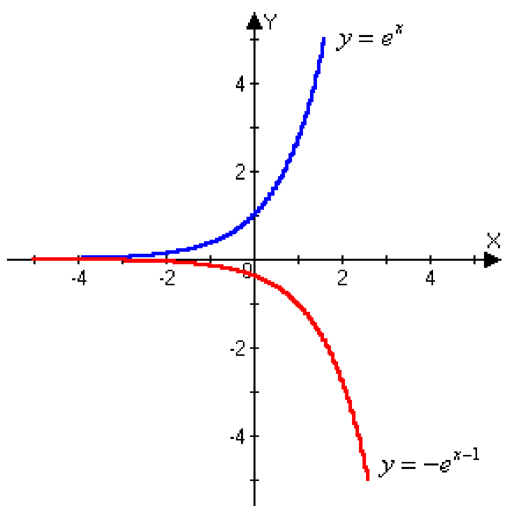
Example 1: Graph the function $y = -e^{x-1}$, not by plotting points, but by starting from the graph of $y = e^x$ in the above figure. State the domain, range, and asymptote.

Solution:

Step 1: We will use transformation techniques to obtain the graph of $y = -e^{x-1}$. Start with the graph of $y = e^x$, reflect it in the *x*-axis and shift it rightward 1 unit.

Step 2: Since our transformation does not involve a vertical shift of the graph, the horizontal asymptote of $y = -e^{x-1}$ is the same as that of $y = e^x$; that is, the horizontal asymptote is the *x*-axis, $y = 0$.

Looking at the graph, we see that the domain of $y = -e^{x-1}$ is all real numbers $(-\infty,\infty)$, and the range is $(-\infty, 0)$.



Continuously compounded interest is calculated by the formula

$$A(t) = Pe^{rt}$$

where	$A(t)$	=	amount after *t* years
	P	=	principal
	r	=	interest rate
	t	=	number of years

Example 2: Find the amount after 7 years if $100 is invested at an interest rate of 13% per year, compounded continuously.

Solution:

Step 1: This problem requires that we find an amount that is compounded continuously, thus we will use the continuously compounded interest formula:

$$A = Pe^{rt}$$

Step 2: The initial amount invested is $100, so $P = 100$. The interest rate is 13% per year, so $r = 0.13$. The amount will be invested for 7 years, so $t = 7$.

Step 3: Now we will substitute the values $P = 100$, $r = 0.13$, and $t = 7$ into the formula for continuously compounded interest.

$$\begin{aligned} A(7) &= 100e^{0.13(7)} \\ &= 248.4322 \end{aligned}$$

Step 4: Thus, the amount after 7 years will be $248.43.

Exponential Models of Population Growth:

Population growth is another application of the exponential function. A population experiencing **exponential growth** increases according to the model

$$n(t) = n_0 e^{rt}$$

where $n(t)$ = population at time t
n_o = initial size of the population
r = relative rate of growth (expressed as a proportion of the population)
t = time

Notice that the formula for population growth is the same as that for continuously compounded interest. In fact, the same principal is at work in both cases: The growth of a population (or an investment) per time period is proportional to the size of the population (or the amount of the investment).

Example 3: The number of bacteria in a culture is given by the function

$$n(t) = 10e^{0.22t}$$

(a) What is the relative rate of growth of this bacterium population? Express your answer as a percentage.
(b) What is the initial population of the culture (at $t = 0$)?
(c) How many bacteria will be in the culture at time $t = 15$?

Solution (a):

Population is modeled using the exponential growth model:

$$n(t) = n_0 e^{rt}$$

where r is the relative rate of growth. By inspecting the equation we are given, we see that $r = 0.22$, or 22%.

Solution (b):

To find the initial population, we find the population at time $t = 0$. We do this by substituting $t = 0$ into the equation for $n(t)$.

$$n(t) = 10e^{0.22t}$$
$$n(0) = 10e^{0.22(0)}$$
$$n(0) = 10(1)$$
$$n(0) = 10$$

Thus, the initial population is 10 bacteria.

Note: We could have solved this problem by inspecting the given equation and noticing that n_o, the initial population size, is 10.

Solution (c):

The number of bacteria at time 15 can be found by substituting $t = 15$ into the equation for $n(t)$.

$$n(t) = 10e^{0.22t}$$
$$n(15) = 10e^{0.22(15)}$$
$$n(15) = 10e^{33}$$
$$n(15) \approx 10(27.1126)$$
$$n(15) \approx 271.1264$$

Thus, the population at time 15 is 271 bacteria.

Example 4: The Ewok population on the planet Endor has a relative growth rate of 3% per year, and it is estimated that the population is 6,500.

(a) Find a function that models the population *t* years from now.
(b) Use the function from part (a) to estimate the Ewok population in 8 years.
(c) Sketch the graph of the population function.

Solution (a):

Step 1: To find a function that models the Ewok population, we will use the exponential growth model

$$n(t) = n_0 e^{rt}$$

To use the model, we will need to determine what the values n_o and r.

Step 2: Since we are not explicitly told when our time starts, we can assume the population was estimated to be 6,500 at time $t = 0$. Thus our initial population is $n_o = 6500$.

Step 3: We are told in the problem that the relative growth rate is 3% per year, so $r = 0.03$.

Step 4: Now we will substitute the values $n_o = 6500$ and $r = 0.03$ into the formula for the exponential growth model to find the function that models the population *t* years from now.

$$n(t) = n_0 e^{rt}$$
$$n(t) = 6500e^{0.03t}$$

Solution (b):

An estimate of the Ewok population in 8 years can be found by substituting $t = 8$ into the equation for $n(t)$.

$$n(t) = 6500e^{0.03t}$$
$$n(8) = 6500e^{0.03(8)}$$
$$n(8) = 6500e^{0.24}$$
$$n(8) \approx 6500(1.2710)$$
$$n(8) \approx 8263.1195$$

Thus, the population in 8 years will be 8263.

Example 4 (Continued):

Solution (c):

Step 1: We will graph the population function $n(t) = 6500e^{0.03t}$ by first making a table of values.

t	$n(t)$
−10	4815.32
0	6500.00
5	7551.92
8	8263.12
20	11843.77
40	21580.76

Step 2: Now we will plot the points found in the previous step, and draw a smooth curve connecting them.