

Exponential Functions and Their Graphs

In this chapter we will study the *exponential function*. These important functions are indispensable in working with problems that involve population growth, decay of radioactive materials, and other processes that occur in nature.

In an exponential function, the function value is obtained by raising a fixed number, called the *base*, to a variable power. Recall that we defined a^x if $a > 0$ and x is a rational number, but we have not yet defined irrational powers. In more advanced texts it is shown that a^x exists for all real number values of x , both rational and irrational. Since this is beyond the scope of our text, we will assume that a^x exists for all real numbers x , and that the *Laws of Exponents are still true when the exponents are real numbers*, without justification.

We begin graphing exponential functions by finding several ordered pairs that belong to the function, plotting these points, and connecting them with a smooth curve.

Example 1: Sketch the graphs of the functions $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ by making a table of values.

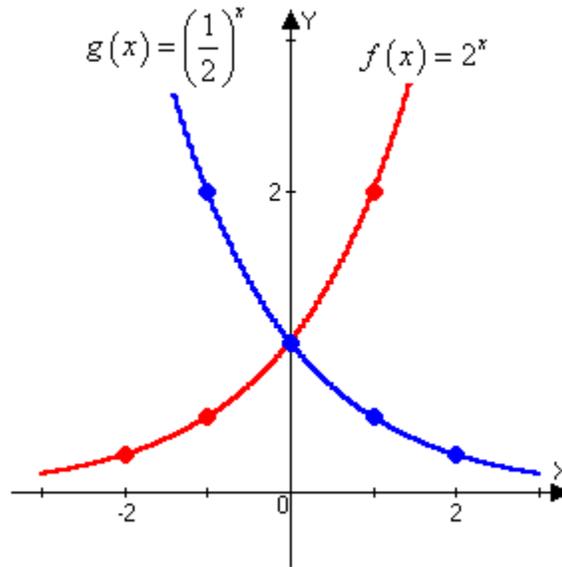
Solution:

Step 1: Choose some values of x and find the corresponding values of $f(x)$ and $g(x)$.

x	$f(x) = 2^x$	$g(x) = \left(\frac{1}{2}\right)^x$
-3	$\frac{1}{8}$	8
-2	$\frac{1}{4}$	4
-1	$\frac{1}{2}$	2
0	1	1
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$

Example 1 (Continued):

Step 2: Plot the points found in the previous step for $f(x)$ and $g(x)$, and draw a smooth curve connecting them.

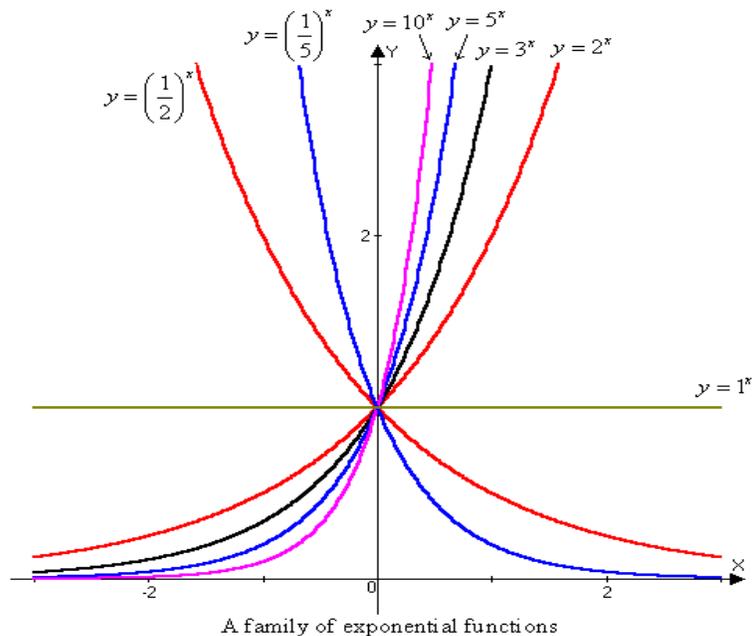


Step 3: Notice that

$$g(x) = \left(\frac{1}{2}\right)^x = \frac{1}{2^x} = 2^{-x} = f(-x)$$

and so we could have obtained the graph of g from the graph of f by reflecting in the y-axis.

The figure below shows the graphs of the family of exponential functions $f(x) = a^x$ for various values of the base a . Notice that all of these graphs pass through the point $(0, 1)$.



From the figure you can see that there are three kinds of exponential functions $y = a^x$:

- If $0 < a < 1$, the exponential function decreases rapidly.
- If $a = 1$, it is constant.
- If $a > 1$, the function increases rapidly, and the larger the base, the more rapid the increase.

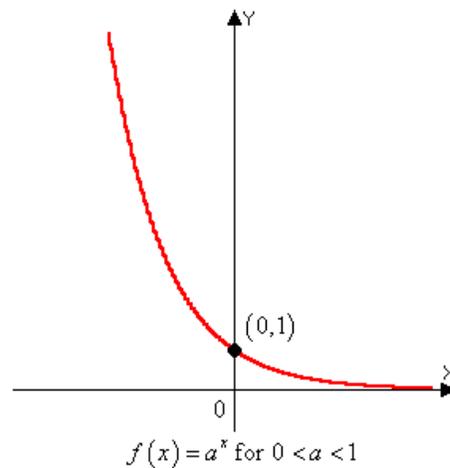
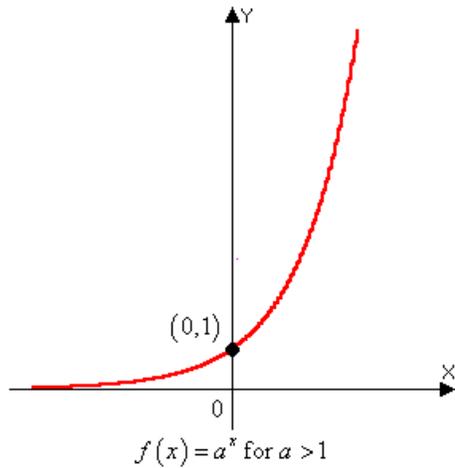
Also, if $a > 1$, then the graph of $y = a^x$ approaches $y = 0$ as x decreases through negative values, and so the x -axis is a horizontal asymptote. If $0 < a < 1$, the graph approaches $y = 0$ as x increases indefinitely and, again, the x -axis is a horizontal asymptote. In either case the graph never touches the x -axis because $a^x > 0$ for all x . Thus, for $a \neq 1$, the exponential function $f(x) = a^x$ has domain $(-\infty, \infty)$ and range $(0, \infty)$. Let's summarize the preceding discussion.

Exponential Functions:

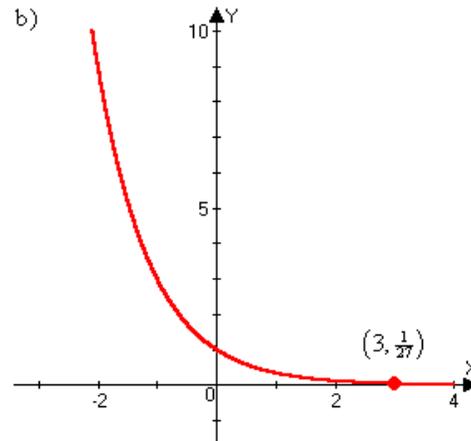
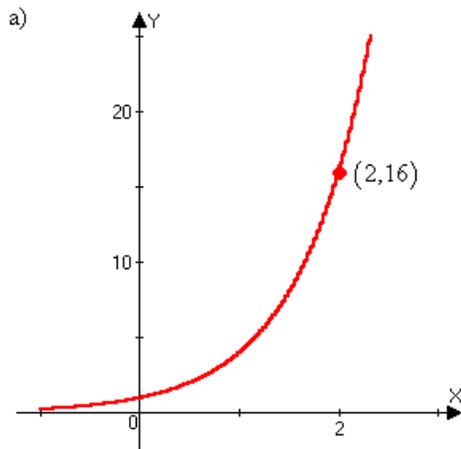
For $a > 0$, the **exponential function with base a** is defined by

$$f(x) = a^x$$

For $a \neq 0$, the domain of f is $(-\infty, \infty)$, the range of f is $(0, \infty)$, and the graph of f has one of the following shapes:



Example 2: Find the exponential function $f(x) = a^x$ whose graph is given.



Solution a):

The function is exponential, so we know the equation will be of the form $f(x) = a^x$. Looking at the graph, we see it contains the point $(2, 16)$. Using this point, we can determine the base of the exponential function, by substituting 2 for x and 16 for $f(x)$.

$$f(x) = a^x$$

$$16 = a^2$$

$$4 = a$$

Thus, the exponential function is $f(x) = 4^x$

Example 2 (Continued):

Solution b):

This graph of an exponential function contains the point $(3, \frac{1}{27})$.

Substituting 3 for x and $\frac{1}{27}$ for $f(x)$ we get

$$f(x) = a^x$$

$$\frac{1}{27} = a^3$$

$$\frac{1}{3} = a$$

Thus the exponential function for this graph is $f(x) = (\frac{1}{3})^x$

In previous sections, we learned how to perform transformations on library functions to find the graphs of more complex functions. We can now add the exponential function to our list of library functions. We perform transformations to the exponential function using the same techniques.

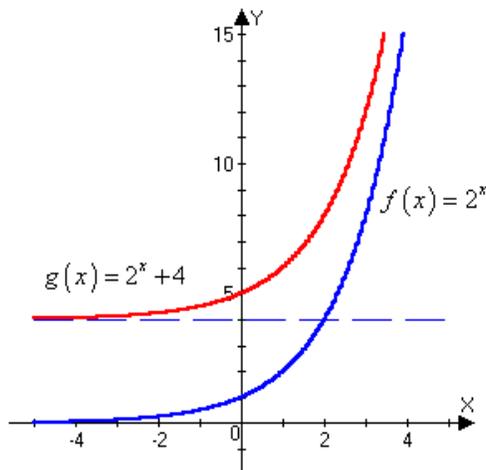
Example 3: Graph the functions, not by plotting points, but by starting from the graphs in the figure found after Example 1 and performing transformations on them. State the domain, range, and asymptote.

a) $g(x) = 2^x + 4$

b) $f(x) = -5^{x-2}$

Solution a):

Step 1: To obtain the graph of $g(x) = 2^x + 4$, we start with the graph of $f(x) = 2^x$ and shift it upward 4 units.

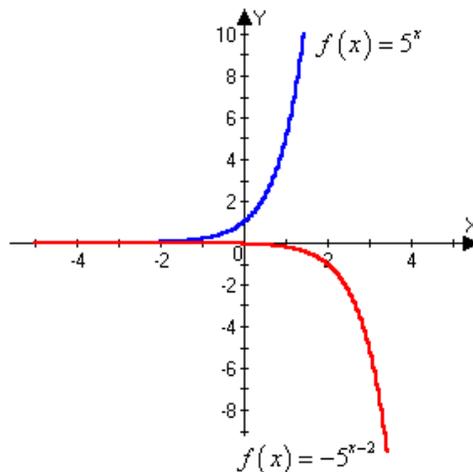


Example 3 (Continued):

Step 2: Notice that while the horizontal asymptote is not actually part of the graph, it also shifts upward 4 units, and so the horizontal asymptote of $g(x) = 2^x + 4$ is the line $y = 4$. Looking at the graph, we see that the domain of g is $(-\infty, \infty)$, and the range is $(4, \infty)$.

Solution b):

Step 1: Based on the translation techniques, the graph of $f(x) = -5^{x-2}$ is the reflection of the graph of $f(x) = 5^x$ in the x -axis, shifted to the right 2 units.



Step 2: Since there was no vertical shift, the horizontal asymptote of f is $y = 0$. Also, the domain is $(-\infty, \infty)$ and the range is $(-\infty, 0)$.

Compound Interest:

If a principal P is invested at an interest rate r for a period of t years, then the amount A of the investment is given by

$$A = P(1 + r) \quad \text{Simple Interest (for one year)}$$

$$A(t) = P(1 + r/n)^{nt} \quad \text{Interest compounded } n \text{ times per year}$$

Example 4: If Sally deposits \$4,000 into an account paying 5% interest compounded quarterly, how much will she have in her account in 3 years (if she does not withdraw any of the money)?

Solution:

Step 1: Identify the given information

Principal (P) = \$4,000
Annual Interest rate (r) = 5% or 0.05
Compounding factor (n) = 4 (quarterly)
Time (t) = 3 years

Step 2: Substitute the known data into the compound interest formula

$$A = P \left(1 + \frac{r}{n} \right)^{nt}$$
$$A = 4000 \left(1 + \frac{0.05}{4} \right)^{4 \cdot 3}$$

Step 3: Simplify

$$A = 4000 \left(1 + \frac{0.05}{4} \right)^{4 \cdot 3}$$
$$A = 4000 (1.0125)^{12}$$
$$A = 4000 (1.1607545\dots)$$
$$A \approx 4643.018$$

Step 4: State your answer

Since we are dealing with money you would round the answer off to the nearest cent. Therefore the answer would be:

Sally would have \$4,643.02 in her account.

Exponential Growth:

If n_0 is the initial size of a population experiencing exponential growth, then the population $n(t)$ at time t is modeled by the function

$$n(t) = n_0 e^{rt}$$

where r is the relative rate of growth expressed as a fraction of the population.

Example 5: Frog population projections

The frog population in a small pond grows exponentially. The current population is 85 frogs, and the relative growth rate is 18% per year.

- (a) Find a function that models the number of frogs after t years.
- (b) Find the projected population after 3 years.

Solution (a): To find the function that models population growth, we need to find the population $n(t)$. To do this, we use the formula for population growth with $n_0 = 85$ and $r = 0.18$. Then

$$n(t) = 85e^{0.18t}$$

Solution (b): We use the population growth function found in (a) with $t = 3$.

$$n(3) = 85e^{0.18(3)}$$

$$n(3) \approx 145.86$$

Use a calculator

Thus, the number of frogs after 3 years is approximately 146.