

Properties of Logarithms

Since the exponential and logarithmic functions with base a are inverse functions, the Properties of Exponents give rise to the Properties of Logarithms.

Properties of Exponents: (a and b are real numbers, m and n are integers)

$$b^0 = 1$$

$$b^1 = b$$

$$(ab)^n = a^n b^n$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

$$\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$$

$$(a^m)^n = a^{mn}$$

$$a^m a^n = a^{m+n}$$

Properties of Logarithms: Let b be a positive number, with $b \neq 1$. Let $M > 0$, $N > 0$, p and x be any real numbers.

$$\log_b 1 = 0$$

$$\log_b b = 1$$

$$\log_b b^x = x$$

$$b^{\log_b x} = x; x > 0$$

$$\log_b MN = \log_b M + \log_b N$$

The logarithm of a product of numbers is the sum of the logarithms of the numbers.

$$\log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N$$

The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.

$$\log_b M^p = p \log_b M$$

The logarithm of a power of a number is the exponent times the logarithm of the number.

Example 1: Use the Properties of Logarithms to rewrite the expression in a form with no logarithm of a product, quotient, or power.

(a) $\log_3 \left(\frac{5x^3}{y^2} \right)$

(b) $\ln \sqrt[5]{x^2(z+1)}$

Solution (a):

$$\begin{aligned} \log_3 \left(\frac{5x^3}{y^2} \right) &= \log_3 (5x^3) - \log_3 y^2 && \text{Law 2} \\ &= \log_3 5 + \log_3 x^3 - \log_3 y^2 && \text{Law 1} \\ &= \log_3 5 + 3\log_3 x - 2\log_3 y && \text{Law 3} \end{aligned}$$

Solution (b):

$$\begin{aligned} \ln \sqrt[5]{x^2(z+1)} &= \ln \left(x^2(z+1) \right)^{\frac{1}{5}} \\ &= \frac{1}{5} \ln \left(x^2(z+1) \right) && \text{Law 3} \\ &= \frac{1}{5} \left[\ln x^2 + \ln(z+1) \right] && \text{Law 1} \\ &= \frac{1}{5} \ln x^2 + \frac{1}{5} \ln(z+1) && \text{Distribution} \\ &= \frac{1}{5} (2 \ln x) + \frac{1}{5} \ln(z+1) && \text{Law 3} \\ &= \frac{2}{5} \ln x + \frac{1}{5} \ln(z+1) && \text{Multiplication} \end{aligned}$$

Example 2: Rewrite the expression as a single logarithm.

(a) $3\log_4 5 + \log_4 10 - \log_4 7$

(b) $\log x - 2\log y + \frac{1}{2}\log(9z^2)$

Example 2 (Continued):

Solution (a):

$$\begin{aligned} 3\log_4 5 + \log_4 10 - \log_4 7 &= \log_4 5^3 + \log_4 10 - \log_4 7 && \text{Law 3} \\ &= \log_4 125 + \log_4 10 - \log_4 7 && \text{Because } 5^3 = 125 \\ &= \log_4 (125 \cdot 10) - \log_4 7 && \text{Law 1} \\ &= \log_4 (1250) - \log_4 7 && \text{Simplification} \\ &= \log_4 \left(\frac{1250}{7} \right) && \text{Law 2} \end{aligned}$$

Solution (b):

$$\begin{aligned} \log x - 2\log y + \frac{1}{2}\log(9z^2) &= \log x - \log y^2 + \log(9z^2)^{\frac{1}{2}} && \text{Law 3} \\ &= \log x - \log y^2 + \log(3z) && \text{Simplification} \\ &= \log\left(\frac{x}{y^2}\right) + \log(3z) && \text{Law 2} \\ &= \log\left[\left(\frac{x}{y^2}\right)(3z)\right] && \text{Law 1} \\ &= \log\left(\frac{3zx}{y^2}\right) && \text{Simplification} \end{aligned}$$

Example 3: Evaluate the expression

- (a) $\log_2 \sqrt[5]{16}$
(b) $\log_3 189 - \log_3 7$

Solution (a):

$$\begin{aligned} \log_2 \sqrt[5]{16} &= \log_2 16^{\frac{1}{5}} \\ &= \frac{1}{5}\log_2 16 && \text{Law 3} \\ &= \frac{1}{5}\log_2 2^4 && \text{Because } 16 = 2^4 \\ &= \frac{1}{5}(4) && \text{Property 3 of Logarithms} \\ &= \frac{4}{5} && \text{Multiplication} \end{aligned}$$

Example 3 (Continued):**Solution (b):**

$$\begin{aligned}
 \log_3 189 - \log_3 7 &= \log_3 \frac{189}{7} \\
 &= \log_3 27 \\
 &= \log_3 3^3 \\
 &= 3
 \end{aligned}$$

Law 2

Simplification

Because $27 = 3^3$

Property 3 of Logarithms

Change of Base:

A calculator can be used to approximate the values of common logarithms (base 10) or natural logarithms (base e). However, sometimes we need to use logarithms to other bases. The following rule is used to convert logarithms from one base to another.

Change of Base Formula:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Example 4: Use the Change of Base Formula and a calculator to evaluate the logarithm, correct to six decimal places. Use either natural or common logarithms.

(a) $\log_6 17$

(b) $\log_5 2.33$

Solution (a):

The Change of Base Formula says

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Thus, if we let the new base $a = 10$

$$\log_6 17 = \frac{\log_{10} 17}{\log_{10} 6} \approx 1.581246$$

Example 4 (Continued):

Solution (b):

Again we will use the Change of Base Formula. This time we will let the new base be $a = e$.

$$\log_5 2.33 = \frac{\ln 2.33}{\ln 5} \approx 0.525568$$

Example 5: Simplify: $(\log_8 12)(\log_{12} 7)$

Solution:

Using the Change of Base Formula with the new base $a = 10$:

$$\log_8 12 = \frac{\log 12}{\log 8} \quad \text{and} \quad \log_{12} 7 = \frac{\log 7}{\log 12}$$

Thus,

$$\begin{aligned} (\log_8 12)(\log_{12} 7) &= \left(\frac{\log 12}{\log 8} \right) \left(\frac{\log 7}{\log 12} \right) \\ &= \frac{\log 7}{\log 8} \\ &= \log_8 7 \end{aligned}$$

Simplification

Change of Base Formula