Logarithmic Functions and Their Graphs

In this section we introduce logarithmic functions. Notice that every exponential function $f(x) = a^x$, with $a > 0$ and $a \neq 1$, is a one-to-one function by the Horizontal Line Test and therefore has an inverse function. The inverse function of the exponential function with base *a* is called the *logarithmic function with base a* and is denoted by $\log_a x$. Recall that f^{-1} is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

This leads to the following definition of the logarithmic function.

Definition of the Logarithmic Function:

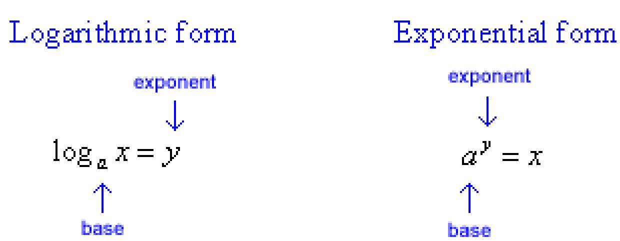
> Let *a* be a positive number with $a \neq 0$. The **logarithmic function with base *a***, denoted by $\mathbf{log_a}$, is defined by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

In other words, this says that

> $\log_a x$ is the exponent to which the base *a* must be raised to give x.

The form $\log_a x = y$ is called the **logarithmic form**, and the form $a^y = x$ is called the **exponential form**. Notice that in both forms the base is the same:



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Example 1: Express each equation in exponential form.

(a) $\log_7 49 = 2$
(b) $\log_{16} 4 = \frac{1}{2}$

Solution:

From the definition of the logarithmic function we know

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

This implies

(a) $\log_7 49 = 2 \quad \Leftrightarrow \quad 7^2 = 49$

(b) $\log_{16} 4 = \frac{1}{2} \quad \Leftrightarrow \quad 16^{\frac{1}{2}} = 4$

Example 2: Express each equation in logarithmic form.

(a) $3^4 = 81$
(b) $6^{-1} = \frac{1}{6}$

Solution:

From the definition of the logarithmic function we know

$$a^y = x \quad \Leftrightarrow \quad \log_a x = y$$

This implies

(a) $3^4 = 81 \quad \Leftrightarrow \quad \log_3 81 = 4$

(b) $6^{-1} = \frac{1}{6} \quad \Leftrightarrow \quad \log_6 \frac{1}{6} = -1$

Graphs of Logarithmic Functions:

Since the logarithmic function $f(x) = log_a\ x$ is the inverse of the exponential function $f(x) = a^x$, the graphs of these two functions are reflections of each other through the line $y = x$.

Also, since the exponential function with a ≠ 0 has domain ℝ and range (0, ∞), we conclude its inverse, the logarithmic function, has domain (0, ∞) and range ℝ. Finally, since $f(x) = a^x$ has a horizontal asymptote at $y = 0$, $f(x) = log_a\ x$ has a vertical asymptote at $x = 0$.

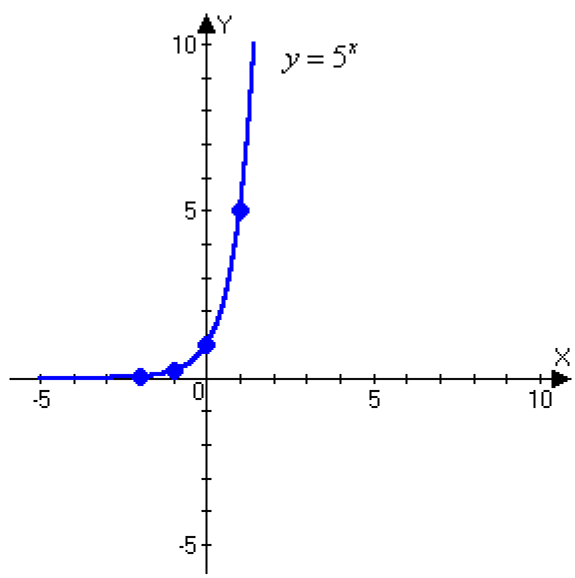
Example 3: Draw the graph of $y = 5^x$, then use it to draw the graph of $y = log_5\ x$.

Solution:

Step 1: To graph $y = 5^x$, start by choosing some values of x and finding the corresponding y-values.

x	y
−2	$\frac{1}{25}$
−1	$\frac{1}{5}$
0	1
1	5
2	25

Step 2: Plot the points found in the previous step for $y = 5^x$ and draw a smooth curve connecting them.

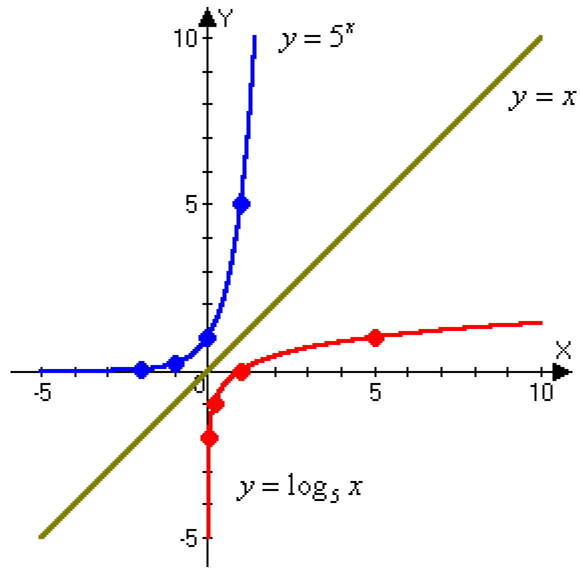


Example 3 (Continued):

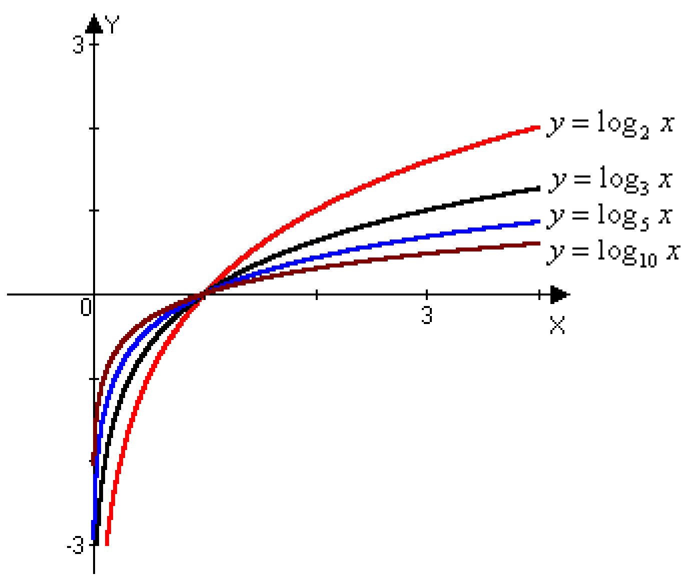
Step 3: To find the graph of $y = log_5\ x$, all we need to do is reflect the graph of $y = 5^x$ over the line $y = x$, because they are inverses.

Another way we can find the graph of $y = log_5\ x$ is to take the chart we found in Step 1 for $y = 5^x$, and switch the x and y values. Then we plot the new points and draw a smooth curve connecting them.

x	y
$\frac{1}{25}$	-2
$\frac{1}{5}$	-1
1	0
5	1
25	2



The figure below shows the graphs of the family of logarithmic functions with bases 2, 3, 5, and 10.

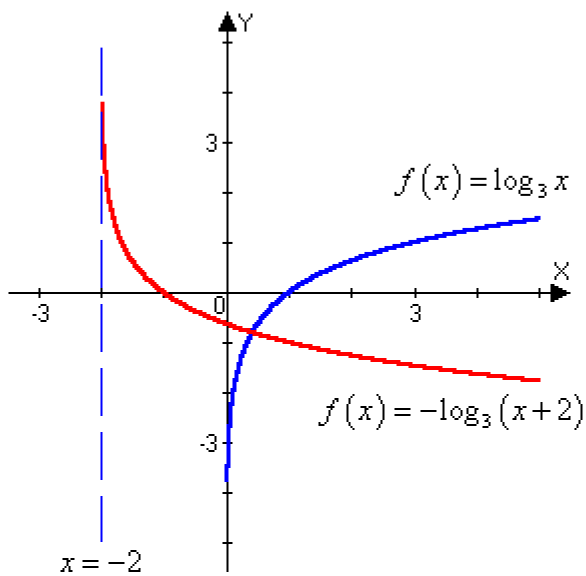


We can now add the logarithmic function to our list of library functions. In addition, we can perform transformations to the logarithmic function using the techniques learned earlier.

Example 4: Graph the function $f(x) = -log_3(x + 2)$, not by plotting points, but by starting from the graphs in the above figure. State the domain, range, and asymptote.

Solution:

Step 1: To obtain the graph of $f(x) = -log_3(x + 2)$, we start with the graph of $f(x) = log_3 x$, reflect it across the x-axis and shift it to the left 2 units.



Step 2: Notice that while the vertical asymptote is not actually part of the graph, it also shifts left 2 units, and so the vertical asymptote of $f(x) = -log_3(x + 2)$ is the line $x = –2$. Looking at the graph, we see that the domain of f is $(–2, \infty)$, and the range is $\mathbb{R}$.

Some important properties of logarithms are as follows:

Properties of Logarithms:

	Property	Reason
1.	$\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2.	$\log_a a = 1$	We must raise a to the power 1 to get a.
3.	$\log_a a^x = x$	We must raise a to the power x to get a^x.
4.	$a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

Common Logarithms:

Frequently one will see the logarithmic function written without a specified base, $y = \log x$. This is known as the common logarithm, and it is the logarithm with base 10.

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\log x = \log_{10} x$$

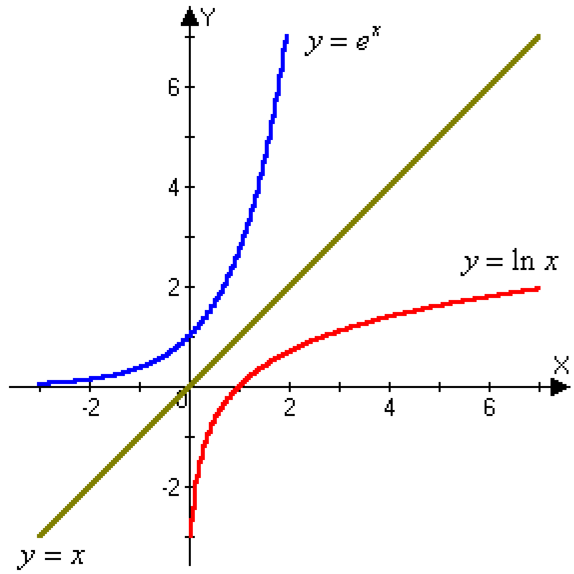
Natural Logarithms:

Of all possible bases a for logarithms, it turns out the most convenient choice for the purposes of calculus is the number e.

The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:

$$\ln x = \log_e x$$

The natural logarithmic function $y = \ln x$ is the inverse function of the exponential function $y = e^x$. Both functions are graphed below.



By the definition of inverse functions we have

$$\ln x = y \quad \Leftrightarrow \quad e^y = x$$

The same important properties of logarithms that were listed above also apply to natural logarithms.

Properties of Natural Logarithms:

	Property	Reason
1.	$\ln 1 = 0$	We must raise e to the power 0 to get 1.
2.	$\ln e = 1$	We must raise e to the power 1 to get e.
3.	$\ln e^x = x$	We must raise e to the power x to get e^x.
4.	$e^{\ln x} = x$	$\ln x$ is the power to which e must be raised to get x.

Example 5: Evaluate the expressions.

(a) $\log_7 1$
(b) $\log_3 3$
(c) $\ln e^{12}$
(d) $10^{\log \pi}$

Example 5 (Continued):

Solution (a):

The first property of logarithms says $\log_a 1 = 0$. Thus,

$$\log_7 1 = 0$$

Solution (b):

The second property of logarithms says $\log_a a = 1$. Thus,

$$\log_3 3 = 1.$$

Solution (c):

The third property of natural logarithms says $\ln e^x = x$. Thus,

$$\ln e^{12} = 12 .$$

Solution (d):

Step 1: First note that $\log \pi = \log_{10} \pi$. So

$$10^{\log \pi} = 10^{\log_{10} \pi}$$

Step 2: The fourth property of logarithms says $a^{\log_a x} = x$. Thus

$$10^{\log_{10} \pi} = \pi \text{.}$$

Example 6: Use the definition of the logarithmic function to find x.

(a) $3 = \log_2 x$
(b) $-4 = \log_3 x$
(c) $4 = \log_x 625$
(d) $-2 = \log_x 100$

Solution (a):

Step 1: By the definition of the logarithm, we can rewrite the expression in exponential form.

$$3 = \log_2 x \quad \Leftrightarrow \quad 2^3 = x$$

Example 6 (Continued):

Step 2: Now we can solve for *x*.

$$x = 2^3$$
$$x = 8$$

Solution (b):

Step 1: Rewrite the expression in exponential form using the definition of the logarithmic function.

$$-4 = \log_3 x \quad \Leftrightarrow \quad 3^{-4} = x$$

Step 2: Solve for *x*.

$$x = 3^{-4}$$
$$x = \tfrac{1}{3^4}$$
$$x = \tfrac{1}{81}$$

Solution (c):

Step 1: Rewrite the expression in exponential form using the definition of the logarithmic function.

$$4 = \log_x 625 \quad \Leftrightarrow \quad x^4 = 625$$

Step 2: Solve for *x*.

$$x^4 = 625 \qquad \text{take the fourth root of both sides}$$
$$x = \pm\sqrt[4]{625}$$
$$x = \pm 5$$

Recall that a logarithm cannot have a negative base. So, we discard the extraneous solution $x = -5$, and therefore $x = 5$ is the only solution to the expression $4 = \log_x 625$.

Example 6 (Continued):

Solution (d):

Step 1: Rewrite the expression in exponential form using the definition of the logarithmic function.

$$-2 = \log_x 100 \quad \Leftrightarrow \quad x^{-2} = 100$$

Step 2: Solve for x.

$$x^{-2} = 100$$

$$\tfrac{1}{x^2} = 100 \qquad \text{multiply both sides by } x^2$$

$$1 = 100x^2 \qquad \text{divide both sides by } 100$$

$$\tfrac{1}{100} = x^2 \qquad \text{take the square root of both sides}$$

$$x = \pm\sqrt{\tfrac{1}{100}}$$

$$x = \pm\tfrac{1}{10}$$

Again we note that a logarithm cannot have a negative base. So, we discard the extraneous solution $x = -\tfrac{1}{10}$, and therefore $x = \tfrac{1}{10}$ is the only solution to the expression $-2 = \log_x 100$.

THE DECIBEL SCALE

The ear is sensitive to an extremely wide range of sound intensities. We take as a reference intensity $I_0 = 10^{-12}$ W/m^2 (watts per square meter) at a frequency of 1000 hertz, which measures a sound that is just barely audible (the threshold of hearing). The psychological sensation of loudness varies with the logarithm of the intensity (the Weber-Fechner Law) and so **intensity level** β, measured in decibels (dB), is defined as

$$\beta = 10\log\left(\frac{I}{I_0}\right)$$

Example 7: The intensity level of the sound of a subway train was measured at 98 dB. Find the intensity in W/m^2.

Solution:

From the definition of intensity level, we see that

$$10\log\left(\frac{I}{10^{-12}}\right)=98$$

$$10\log(I)-10\log(10^{-12})=98 \qquad \text{Law 2 of Logarithms}$$

$$10\log(I)=98+10\log(10^{-12}) \qquad \text{Add } 10\log(10^{-2}) \text{ to both sides}$$

$$\log(I)=9.8+\log(10^{-12}) \qquad \text{Divide both sides by 10}$$

$$\log(I)=9.8-12=-2.2 \qquad \text{Property of logarithms}$$

$$I=10^{-2.2} \qquad \text{Property of logarithms}$$

$$I\approx 6.31\times 10^{-3} \qquad \text{Use a calculator}$$

Thus, the intensity level is about 6.3×10^{-3} W/m^2.

THE RICHTER SCALE

In 1935 the American geologist Charles Richter (1900-1984) defined the magnitude M of an earthquake to be

$$M=\log\left(\frac{I}{S}\right)$$

where I is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake) and S is the intensity of a “standard” earthquake (whose amplitude is 1 micron = 10^{-4} cm). The magnitude of a standard earthquake is

$$M=\log\left(\frac{S}{S}\right)=\log(1)=0$$

Richter studied many earthquakes that occurred between 1900 and 1950. The largest had magnitude 8.9 on the Richter scale, and the smallest had a magnitude 0. This corresponds to a ratio of intensities of 800,000,000, so the Richter scale provides more manageable numbers to work with. For instance, an earthquake of magnitude 6 is ten times stronger than an earthquake of magnitude 5.

Example 8: The 1985 Mexico City earthquake had a magnitude of 8.1 on the Richter scale. The 1976 earthquake in Tangshan, China, was 1.26 times as intense. What was the magnitude of the Tangshan earthquake?

Solution:

If I is the intensity of the Mexico City earthquake, then from the definition of magnitude we have

$$M = \log\left(\frac{I}{S}\right) = 8.1$$

The intensity of the Tangshan earthquake was $1.26I$, so its magnitude was

$$M = \log\left(\frac{1.26I}{S}\right) = \log(1.26) + \log\left(\frac{I}{S}\right) = \log(1.26) + 8.1 \approx 8.2$$

Example 9: The 1906 earthquake in San Francisco had a magnitude of 8.3 on the Richter scale. At the same time in Japan an earthquake with magnitude 4.9 caused only minor damage. How many times more intense was the San Francisco earthquake than the Japanese earthquake?

Solution:

If I_1 and I_2 are the intensities of the San Francisco and Japan earthquakes, then we are required to find I_1 / I_2. To relate this to the definition of magnitude, we divide numerator and denominator by S.

$$\log\frac{I_1}{I_2} = \log\frac{I_1/S}{I_2/S}$$ Divide numerator and denominator by S

$$= \log\frac{I_1}{S} - \log\frac{I_2}{S}$$ Law 2 of Logarithms

$$= 8.3 - 4.9 = 3.4$$ Definition of earthquake magnitude

Therefore

$$\frac{I_1}{I_2} = 10^{\log(I_1/I_2)} = 10^{3.4} \approx 2{,}511.88$$

The San Francisco earthquake was about 2500 times as intense as the Japan earthquake.