Quadratic Functions

In this section we begin the study of functions defined by polynomial expressions. Polynomial and rational functions are the most common functions used to model data, and are used extensively in mathematical models of production costs, consumer demands, wildlife management, biological processes, and many other scientific studies. Using these functions and their graphs, predictions regarding future trends can be made.

Polynomial Functions and Their Graphs:

Before we start looking at polynomials, we should know some common terminology.

Definition: A **polynomial of degree *n*** is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x^1 + a_0$$

where $a_n \neq 0$. The numbers $a_0, a_1, a_2, ..., a_n$ are called the **coefficients** of the polynomial. The number a_0 is the **constant coefficient** or **constant term**. The number a_n, the coefficient of the highest power is the **leading coefficient**, and the term $a_n x^n$ is the **leading term**.

Notice that a polynomial is usually written in descending powers of the variable, and the degree of a polynomial is the power of the leading term. For instance

$$P(x) = 4x^3 - x^2 + 5$$

is a polynomial of degree 3. Also, if a polynomial consists of just a single term, such as $Q(x) = 7x^4$, then it is called a **monomial**.

Polynomials of degree 0 are constant functions and polynomials of degree 1 are linear functions, whose graphs are both straight lines. Polynomials of degree 2 are quadratic equations, and their graphs are parabolas. As the degree of the polynomial increases beyond 2, the number of possible shapes the graph can be increases. However, the graph of a polynomial function is always a smooth continuous curve (no breaks, gaps, or sharp corners).

Monomials of the form $P(x) = x^n$ are the simplest polynomials.

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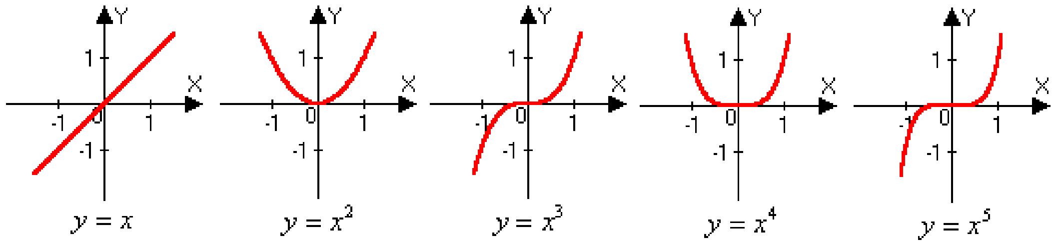


As the figure suggest, the graph of $P(x) = x^n$ has the same general shape as $y = x^2$ when *n* is even, and the same general shape as $y = x^3$ when *n* is odd. However, as the degree *n* becomes larger, the graphs become flatter around the origin and steeper elsewhere.

To this point, we have had some experience with quadratic equations. We know that the graph of a quadratic equation gives us a *parabola*. In this section, we will see how quadratic equations (and their corresponding parabolas) can give us some valuable information in real-life situations.

If a parabola "opens upward," then we know that there is a point at which the graph reaches its minimum value. Likewise, if we have a parabola that "opens downward," we know that there must be a maximum value for that function. These maximum and minimum values are referred to as *extreme values.*

There are many practical situations, which can be represented by such functions. In these situations, the extreme values can often give us valuable pieces of information. For example, the owner of a clothing store might be interested in knowing her maximum profit or minimum cost.

As review, we will look at the definition of a quadratic function.

Def: A **quadratic function** is a function *f* of the form

$$f(x) = ax^2 + bx + c$$

where *a*, *b*, and *c* are real numbers and $a \neq 0$.

Standard Form and Completing the Square:

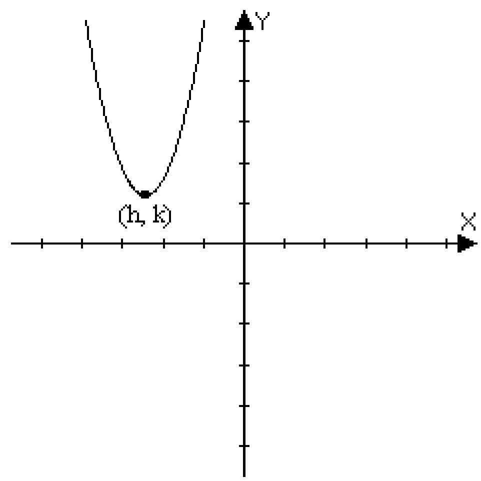
In determining extreme values, it is very helpful to put the quadratic function into standard form. This can be done by *completing the square*. Now, I can already hear the collective groan at the mention of "completing the square," but it is very necessary skill in determining extreme values. I am confident that you can learn it well if you just take it step-by-step. First, let's take a look at standard form.

Def: A quadratic function can be expressed in the **standard form**

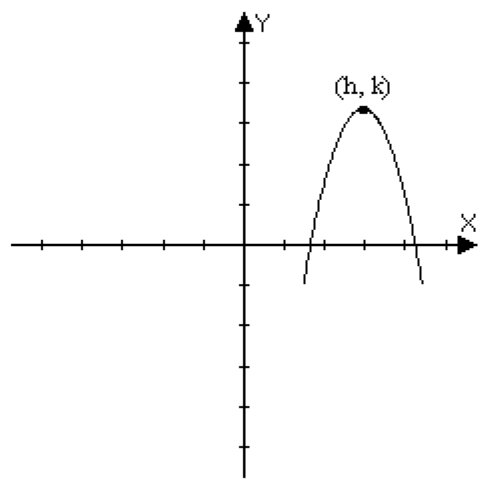
$$f(x) = a(x-h)^2 + k$$

by completing the square. The graph of *f* is a parabola with **vertex** *(h,k)*; the parabola opens upward if $a > 0$ or downward if $a < 0$.

If $a > 0$, then the **minimum value** of *f* occurs at $x = h$ and this value is $f(h) = k$, in other words, the point with coordinates *(h,k)*.



If $a < 0$, then the **maximum value** of *f* occurs at $x = h$ and this value is $f(h) = k$, also the point *(h,k)*.



Example 1: Sketch the graph of the function $f(x)=x^2+10x-20$ and state the vertex. Begin by completing the square.

Solution: Comparing this with the form $f(x)=ax^2+bx+c$, we can see that

$a=1$

$b=10$

$c=-20$

Example 1 (Continued):

Step 1: Group the x^2 and *x* terms, and factor out any *a* value.

In this case, since *a* = 1, we simply have to group the first two terms.

$f(x)=(x^2+10x)-20$

Step 2: *After* grouping the first two terms and factoring out *a*, consider the coefficient of *x*, in this case 10.

Divide this value by 2 and take the square.

$$\left(\frac{10}{2}\right)^2=5^2=25$$

Step 3: Complete the square by *adding* the above value *inside* the parentheses.

<u>*Note:*</u> Any value that is added to one side of the equation must also be subtracted from that side of the equation in order to keep from changing the value. *i.e.*- add a value of zero to the equation.

Therefore, we must *subtract* the total value *outside* the parentheses. (Here is where you must be careful to pay attention to any common value that has been factored out. We will see this in the next example.)

$f(x)=(x^2+10x+25)-20-25$

Step 4: Factor the portion of the equation in parentheses and combine any other like terms.

$f(x)=(x^2+10x+25)-20-25$

$=(x^2+10x+25)-45$

$=(x+5)^2-45$

<u>**OR**</u> $f(x)=(x-(-5))^2-45$ (which will be helpful, as you will see now)

The function should be in the form $f(x) = a(x-h)^2 + k$.

In this case,

$a = 1$

$h = -5$

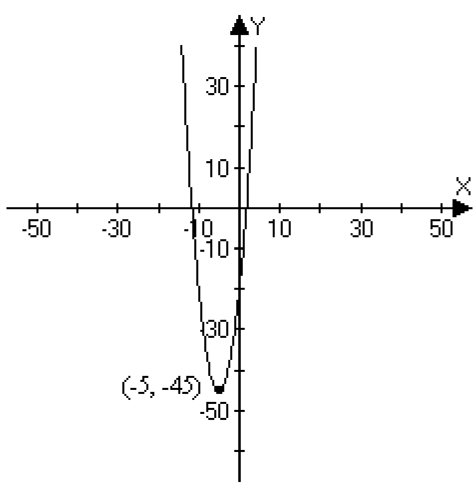
$k = -45$

Example 1 (Continued):

Notice: You must pay special attention to the sign of *h*. Rewriting the equation as we did above will be helpful in seeing the sign of *h* more clearly.

Now that we have completed the square, the remainder of the solution is very simple.

By definition, we know the parabola opens upward, since *a* > 0, and we have a minimum value at the point (-5,-45) which is also the vertex of the parabola. We are simply asked to sketch the parabola, so it is not necessary in this case to find multiple points on the graph. We can simply sketch the graph using what we know.



Let's look at one more example.

Example 2: For the function $f(x) = 2x^2 - 16x + 12$

(a) Express the function in standard form.
(b) Sketch its graph.
(c) Find its maximum or minimum value.

Solution:

(a) We can put the function into standard form by completing the square.

Step 1: Group the x^2 and *x* terms, and factor out any *a* value.

$$f(x) = (2x^2 - 16x) + 12$$
$$= 2(x^2 - 8x) + 12$$

Example 2 (Continued):

Step 2: *After* grouping the first two terms and factoring out *a*, consider the coefficient of *x*, in this case (–8).

Divide this value by 2 and take the square.

$$\left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$$

(Notice that the sign of the *x* coefficient does not really matter at this point, because after squaring the term, you will always have a positive number.)

Step 3: Complete the square by *adding* the above value *inside* the parentheses and *subtracting* the total value *outside* the parentheses.

$$f(x) = 2(x^2 - 8x + 16) + 12 - 32$$

Note: Special attention must be given to the value that was factored out. This will affect the value which is subtracted outside the parentheses. *i.e.*-2(16)=32

Step 4: Factor the portion of the equation in parentheses and combine any other like terms.

$$f(x) = 2(x^2 - 8x + 16) + 12 - 32$$
$$= 2(x^2 - 8x + 16) - 20$$
$$= 2(x - 4)^2 - 20$$

So, we have the graph in standard form.

(b) To sketch the graph, let's combine the information we have from the standard form of the equation.

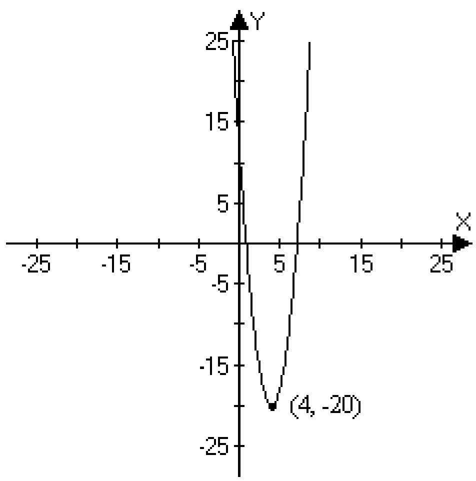
$a = 2$

$h = 4$

$k = -20$

Since $a > 0$, the parabola opens upward; therefore, we have a minimum value at $x = 4$. This value occurs at the point (4,-20) which is also the vertex. With this information, we can easily sketch the graph.

Example 2 (Continued):



(c) The minimum value of the graph occurs at $x = 4$, and the value there is (–20). This is the value $f(h) = k$.

Maximum or Minimum Value of a Quadratic Equation:

In the examples so far, we have been asked to graph the function, etc. Sometimes is simply necessary to know the maximum or minimum value. In this case, you do not need to complete the square, because you do not need the equation to be in standard form.

Def: The maximum or minimum value of a quadratic function $f(x) = ax^2 + bx + c$ occurs at

$$x = -\frac{b}{2a}$$

If $a > 0$, then the **minimum value** is $f\left(-\frac{b}{2a}\right)$.

If $a < 0,$ then the **maximum value** is $f\left(-\frac{b}{2a}\right).$

Example 3: A coffee shop on the SAC campus has a sales analyst consider the store's sales records for a month. The analyst produces a formula in order to help the store generate more profit. He finds that if the shop cells *l* orders of latte in one day, the profit (in dollars) is given by the following function

$$P(l) = -\frac{1}{1000}x^2 + 3x - 1800$$

What is the maximum profit possible for the store in one day, and how many lattes must they sell in order to reach this maximum?

Solution: First, it is important to notice that they asked for a *maximum* profit. We know we will have a maximum instead of a minimum, because $a < 0$. Using the definition above, the maximum occurs at

$$l = -\frac{b}{2a} = -\frac{3}{2\left(-\frac{1}{1000}\right)} = -\frac{3(1000)}{2(-1)} = \frac{3000}{2} = 1500$$

This means that in order to receive a maximum profit, the coffee shop must sell 1500 lattes per day.

In order to find out what the maximum profit actually is (in dollars), we have to evaluate the function at this value, *l*.

$$P\left(-\frac{b}{2a}\right) = P(1500) = -\frac{1}{1000}(1500)^2 + 3(1500) - 1800$$

$$= -\frac{2250000}{1000} + 4500 - 1800$$

$$= -2250 + 4500 - 1800$$

$$= 450$$

The maximum profit would be \$450.