

Properties and Tests of Zeros of Polynomial Functions

The Remainder and Factor Theorems:

Synthetic division can be used to find the values of polynomials in a sometimes easier way than substitution. This is shown by the next theorem.

If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

We learned that if c is a *zero* of P , then $x - c$ is a *factor* of $P(x)$. The next theorem restates this fact in a more useful way.

Factor Theorem: c is a zero of P if and only if $x - c$ is a factor of $P(x)$.

An important consequence of the Factor Theorem is that finding the zeros of a polynomial is really the same thing as factoring it into linear factors. In this section we will study more methods that help us find the real zeros of a polynomial, and thereby factor the polynomial.

Rational Zeros of Polynomials:

The next theorem gives a method to determine all possible candidates for rational zeros of a polynomial function with integer coefficients.

Rational Zeros Theorem:

If the polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ has integer coefficients, then every rational zero of P is of the form

$$\frac{p}{q}$$

where p is a factor of the constant coefficient a_0

and q is a factor of the leading coefficient a_n

Example 1: List all possible rational zeros given by the Rational Zeros Theorem of $P(x) = 6x^4 + 7x^3 - 4$ (but don't check to see which actually are zeros) .

Solution:

Step 1: First we find all possible values of p , which are all the factors of $a_0 = 4$. Thus, p can be ± 1 , ± 2 , or ± 4 .

Example 1 (Continued):

Step 2: Next we find all possible values of q , which are all the factors of $a_n = 6$. Thus, q can be $\pm 1, \pm 2, \pm 3$, or ± 6 .

Step 3: Now we find the possible values of $\frac{p}{q}$ by making combinations of the values we found in Step 1 and Step 2. Thus, $\frac{p}{q}$ will be of the form $\frac{\text{factors of 4}}{\text{factors of 6}}$. The possible $\frac{p}{q}$ are

$$\pm \frac{1}{1}, \pm \frac{2}{1}, \pm \frac{4}{1}, \pm \frac{1}{2}, \pm \frac{2}{2}, \pm \frac{4}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{1}{6}, \pm \frac{2}{6}, \pm \frac{4}{6}$$

Step 4: Finally, by simplifying the fractions and eliminating duplicates, we get the following list of possible values for $\frac{p}{q}$.

$$\pm 1, \pm 2, \pm 4, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{1}{6}$$

Now that we know how to find all possible rational zeros of a polynomial, we want to determine which candidates are actually zeros, and then factor the polynomial. To do this we will follow the steps listed below.

Finding the Rational Zeros of a Polynomial:

1. **Possible Zeros:** List all possible rational zeros using the Rational Zeros Theorem.
2. **Divide:** Use Synthetic division to evaluate the polynomial at each of the candidates for rational zeros that you found in Step 1. When the remainder is 0, note the quotient you have obtained.
3. **Repeat:** Repeat Steps 1 and 2 for the quotient. Stop when you reach a quotient that is quadratic or factors easily, and use the quadratic formula or factor to find the remaining zeros.

Synthetic Division:

Synthetic division is a shortcut method of performing long division that can be used when the divisor is a first degree polynomial of the form $x - c$. In synthetic division we write only the essential part of the long division table. To illustrate, compare these long division and synthetic division tables, in which we divide $3x^3 - 4x + 2$ by $x - 1$:

Long Division

$$\begin{array}{r}
 \overline{) 3x^3 + 0x^2 - 4x + 2} \\
 \underline{3x^3 - 3x^2} \\
 3x^2 - 4x \\
 \underline{3x^2 - 3x} \\
 -x + 2 \\
 \underline{-x + 1} \\
 1
 \end{array}$$

← remainder

Synthetic Division

$$\begin{array}{r|rrrr}
 1 & 3 & 0 & -4 & 2 \\
 & & 3 & 3 & -1 \\
 \hline
 & 3 & 3 & -1 & 1
 \end{array}$$

↑ quotient ↑ remainder

Note that in synthetic division we abbreviate $3x^3 - 4x + 2$ by writing only the coefficients: 3 0 -4 2, and instead of $x - 1$, we simply write 1. (Writing 1 instead of -1 allows us to add instead of subtract, but this changes the sign of all the numbers that appear in the yellow boxes.)

To divide $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ by $x - c$, we proceed as follows:

$$\begin{array}{r|rrrrrrrr}
 c & a_n & a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_2 & a_1 & a_0 \\
 & & cb_{n-1} & cb_{n-2} & cb_{n-3} & \dots & cb_2 & cb_1 & cb_0 \\
 \hline
 & & b_{n-1} & b_{n-2} & b_{n-3} & b_{n-4} & \dots & b_1 & b_0 & r
 \end{array}$$

Here $b_{n-1} = a_n$, and each number in the bottom row is obtained by adding the numbers above it. The remainder is r and the quotient is

$$b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_1 x + b_0$$

Example 2: Find the quotient and the remainder of $\frac{x^4 - 7x^2 - 6x}{x + 2}$ using synthetic division.

Solution:

Step 1: We put $x + 2$ in the form $x - c$ by writing it as $x - (-2)$. Use this and the coefficients of the polynomial to obtain

$$-2 \left| \begin{array}{ccccc} 1 & 0 & -7 & -6 & 0 \end{array} \right.$$

Note that we used 0 as the coefficient of any missing powers.

Step 2: Next, bring down the 1.

$$\begin{array}{r} -2 \left| \begin{array}{ccccc} 1 & 0 & -7 & -6 & 0 \end{array} \right. \\ \hline 1 \end{array}$$

Step 3: Now, multiply -2 by 1 to get -2 , and add it to the 0 in the first row. The result is -2 .

$$\begin{array}{r} -2 \left| \begin{array}{ccccc} 1 & 0 & -7 & -6 & 0 \end{array} \right. \\ \quad \quad \quad -2 \\ \hline 1 \quad -2 \end{array}$$

Step 4: Next, $-2(-2) = 4$. Add this to the -7 in the first row.

$$\begin{array}{r} -2 \left| \begin{array}{ccccc} 1 & 0 & -7 & -6 & 0 \end{array} \right. \\ \quad \quad \quad -2 \quad 4 \\ \hline 1 \quad -2 \quad -3 \end{array}$$

Example 2 (Continued):

Step 5: $-2(-3) = 6$. Add this to the -6 in the first row.

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -7 & -6 & 0 \\ & & -2 & 4 & 6 & \\ \hline & 1 & -2 & -3 & 0 & \end{array}$$

Step 6: Finally, $-2(0) = 0$, which is added to 0 to get 0.

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -7 & -6 & 0 \\ & & -2 & 4 & 6 & 0 \\ \hline & 1 & -2 & -3 & 0 & 0 \end{array}$$

The coefficients of the quotient polynomial and the remainder are read directly from the bottom row. Also, the degree of the quotient will always be one less than the degree of the dividend.

Thus, $Q(x) = x^3 - 2x^2 - 3x$ and $R(x) = 0$.

Example 3: Find all real zeros of the polynomial $P(x) = 2x^4 + x^3 - 6x^2 - 7x - 2$.

Solution:

Step 1: First list all possible rational zeros using the Rational Zeros Theorem. For the rational number $\frac{p}{q}$ to be a zero, p must be a factor of $a_0 = 2$ and q must be a factor of $a_n = 2$. Thus the possible rational zeros, $\frac{p}{q}$, are

$$\pm 1, \pm 2, \pm \frac{1}{2}$$

Step 2: Now we will use synthetic division to evaluate the polynomial at each of the candidates for rational zeros we found in Step 1. When we get a remainder of zero, we have found a zero.

Example 3 (Continued):

$$\begin{array}{r|rrrrr} 1 & 2 & 1 & -6 & -7 & -2 \\ & & 2 & 3 & -3 & -10 \\ \hline & 2 & 3 & -3 & -10 & -12 \end{array} \leftarrow \text{Since the remainder is not zero,} \\ \text{+1 is not a zero}$$

$$\begin{array}{r|rrrrr} -1 & 2 & 1 & -6 & -7 & -2 \\ & & -2 & 1 & 5 & 2 \\ \hline & 2 & -1 & -5 & -2 & 0 \end{array} \leftarrow \text{Since the remainder is zero,} \\ \text{-1 is a zero}$$

This also tells us that P factors as

$$2x^4 + x^3 - 6x^2 - 7x - 2 = (x + 1)(2x^3 - x^2 - 5x - 2)$$

Step 3: We now repeat the process on the quotient polynomial $2x^3 - x^2 - 5x - 2$. Again using the Rational Zeros Theorem, the possible rational zeros of this polynomial are

$$\pm 1, \pm 2, \pm \frac{1}{2}.$$

Since we determined that $+1$ was not a rational zero in Step 2, we do not need to test it again, but we should test -1 again.

$$\begin{array}{r|rrrr} -1 & 2 & -1 & -5 & -2 \\ & & -2 & 3 & 2 \\ \hline & 2 & -3 & -2 & 0 \end{array} \leftarrow \text{Since the remainder is zero,} \\ \text{-1 is again a zero}$$

Thus, P factors as

$$\begin{aligned} 2x^4 + x^3 - 6x^2 - 7x - 2 &= (x + 1)(2x^3 - x^2 - 5x - 2) \\ &= (x + 1)(x + 1)(2x^2 - 3x - 2) \\ &= (x + 1)^2(2x^2 - 3x - 2) \end{aligned}$$

Example 3 (Continued):

Step 4: At this point the quotient polynomial, $2x^2 - 3x - 2$, is quadratic. This factors easily into $(x - 2)(2x + 1)$, which tells us we have zeros at $x = 2$ and $x = -\frac{1}{2}$, and that P factors as

$$\begin{aligned} 2x^4 + x^3 - 6x^2 - 7x - 2 &= (x + 1)(2x^3 - x^2 - 5x - 2) \\ &= (x + 1)(x + 1)(2x^2 - 3x - 2) \\ &= (x + 1)^2(2x^2 - 3x - 2) \\ &= (x + 1)^2(x - 2)(2x + 1) \end{aligned}$$

Step 5: Thus the zeros of $P(x) = 2x^4 + x^3 - 6x^2 - 7x - 2$ are $x = -1$, $x = 2$, and $x = -\frac{1}{2}$.

Fundamental Theorem of Algebra:

In 1799 the German mathematician C. F. Gauss proved the Fundamental Theorem of Algebra. This Theorem forms the basis for much of our work in factoring polynomials and solving polynomial equations.

Fundamental Theorem of Algebra:

Every polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ ($n \geq 1, a_n \neq 0$) with complex coefficients has at least one complex zero.

Because any real number is also a complex number, the theorem applies to polynomials with real coefficients as well.

The conclusion of the Fundamental Theorem of Algebra is that for every polynomial $P(x)$, there is a complex number c_1 such that

$$P(c_1) = 0.$$

From the Factor Theorem (Section 5.2), this tells us $x - c_1$ is a factor of $P(x)$. Thus we can write

$$P(x) = (x - c_1)Q(x)$$

where $Q(x)$ has degree $n - 1$. If the quotient $Q(x)$ has degree ≥ 1 we can repeat the procedure of obtaining a factor and a quotient with degree 1 less than the previous quotient until we arrive at the complete factorization of $P(x)$. This process is summarized by the next theorem.

Complete Factorization Theorem:

If $P(x)$ is a polynomial of degree $n > 0$, then there exist complex numbers a, c_1, c_2, \dots, c_n (with $a \neq 0$) such that

$$P(x) = a(x - c_1)(x - c_2) \dots (x - c_n)$$

In the Complete Factorization Theorem the numbers c_1, c_2, \dots, c_n are the zeros of P . These zeros need not all be different. If the factor $x - c$ appears k times in the complete factorization of $P(x)$, then we say that c is a zero of **multiplicity k** .

The next theorem follows from the Complete Factorization Theorem.


Zeros Theorem:

Every polynomial of degree $n \geq 1$ has exactly n zeros, provided that a zero of multiplicity k is counted k times.

Descartes' Rule of Signs:

In many cases, we will have a lengthy list of possible rational zeros of a polynomial. A theorem that is helpful in eliminating candidates is Descartes' Rule of Signs.

In the theorem, **variation in sign** is a change from positive to negative, or negative to positive in successive terms of the polynomial. Missing terms (those with 0 coefficients) are counted as no change in sign and can be ignored. For example,

$$P(x) = 3x^3 + x^2 - 15x + 2$$


has two variations in sign.

Descartes' Rule of Signs: Let P be a polynomial with real coefficients


1. The number of positive real zeros of $P(x)$ is either equal to the number of variations in sign in $P(x)$ or is less than that by an even whole number.
2. The number of negative real zeros of $P(x)$ is either equal to the number of variations in sign in $P(-x)$ or is less than that by an even whole number.

Example 4: Use Descartes' Rule of Signs to determine how many positive and how many negative real zeros $P(x) = 6x^3 + 17x^2 - 31x - 12$ can have. Then determine the possible total number of real zeros.

Solution:

Step 1: First we will count the number of variations in sign of

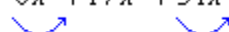
$$P(x) = 6x^3 + 17x^2 - 31x - 12.$$

$$P(x) = 6x^3 + 17x^2 - 31x - 12$$


Since there is only one variation, $P(x)$ has one positive real zero.

Step 2: Now we will count the number of variations in sign of

$$P(-x) = -6x^3 + 17x^2 + 31x - 12.$$

$$P(-x) = -6x^3 + 17x^2 + 31x - 12$$


$P(-x)$ has two variations in sign, thus $P(x)$ has two or zero negative real zeros.

Step 3: Finally by combining our findings in the previous steps, $P(x)$ has either one or three real zeros.

Positive	Negative	Real
1	0	1
1	2	3